Distance Regular Cayley Graphs

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Abstract

Let G be a group and let S be a subset of G. A *Cayley graph*, Cay(S;G), is a graph with vertex set G and edge set $\{(x, xs) : x \in G, s \in S\}$. A graph H is *distance regular* if, given a pair of vertices x and y of distance i apart, $P^i_{j,k}(x, y)$, the number of vertices z of distance j from x and distance k from y, depends only on the values of i, j and k and is independent of the choices for x and y.

We find distance regular Cayley graphs of diameter two and three using the eigenvalues of the adjacency matrix of the distance regular graph and the characters of the group G. To find such graphs we begin with a group G and the parameters for a distance regular graph and build a set S which will yield a Cayley graph.

We also look into methods for proving the nonexistence of distance regular Cayely graphs.

Next, we look at two methods of constructing distance regular graphs. For the first method, we begin with a set S consisting of a number of basic subgroups of G and show that the graph is also distance regular. Next, we begin with a specific type of distance regular graph of a certain diameter and fuse the first and last neighborhoods of an arbitrary vertex and show that this produces a distance regular Cayley graph of smaller diameter.

Contents

1 Preliminaries

We will begin with some basic definitions. First, we will introduce the notion of a group. Next, we will define and give and example of both a Cayley graph and a distance regular graph.

1.1 Groups

Definition 1: A group is a set, G, with a binary operation, $\star$, such that the following properties hold for all $a, b, c \in G$:

1) $a \star b \in G$ (Closure)

2) $a \star (b \star c) = (a \star b) \star c$ (Associativity)

3) There exists an element $e \in G$ such that $a \star e = e \star a = a$ (Identity)

4) For every element $a \in G$, there exists an element $a^{-1} \in G$ such that $a \star a^{-1} = a^{-1} \star a = e$ (Inverse)

Some examples of groups include the integers under addition, nonzero rationals under multiplication, and permutations under function composition.

Note: We will be dealing with *abelian groups*, which are groups which have the commutative property; that is, $a \star b = b \star a$.

Another note: We will be using *direct products* of groups. The notation for a direct product of groups is:

$$G_1 \times G_2 \times ... \times G_n = \{(g_1, g_2, ..., g_n)|\ g_1 \in G_1, g_2 \in G_2,\ ...\ , g_n \in G_n\}$$

where

$$(g_1,\ g_2,\ ...\ ,\ g_n) \star (g_1',\ g_2', \ddots ,'_n) = (g_1 \star_1 g_1',\ g_2 \star_2 g_2',\ ...\ ,\ g_n \star_n g_n')$$

and $\star_i$ is the binary operation for G_i.

Specifically, we will be using direct products of the group:

$$C_n = \{x|\ x^n = 1\}$$

where we are changing to a multiplicative notation.

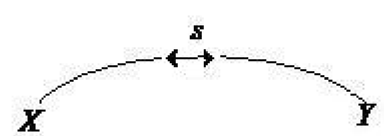
1.2 Cayley Graphs

Assume we have a group G and a subset $S \subseteq G$.

Definition 2: A *Cayley graph*, $\mathrm{Cay}(S; G)$, is a graph whose vertices are the elements of G and whose edges are defined by the adjacency that for all pairs of vertices x and y, $x \sim y$ if $yx^{-1} \in S$.

But $yx^{-1} \in S \Longrightarrow yx^{-1} = s \Longrightarrow y = sx$ for some $s \in S$.

So $V = G$ and $E = \{(x, sx)|\ x \in G$ and $s \in S\}$.



Note: For our purpose we will have some additional requirements for the set S. To avoid loops and multiple edges, that is, to get a simple graph, we require that the identity, 1, not be an element of S and require that S be closed under inverses.

An example of a Cayley Graph

Let our group be:

$$G = C_2 \times C_2 \times C_2 = \{(x, y, z)|\ x^2 = y^2 = z^2 = 1\}$$

.

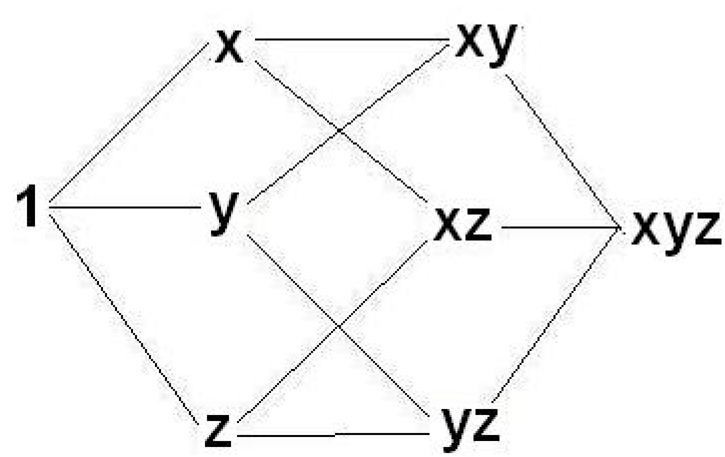
and let our set S be the set:

$$S = \{x, y, z\}$$

But if we write our set S as a sum of subgroups or group elements, then we can write set S as:

$$S = (< x > -1) + (< y > -1) + (< z > -1) = x + y + z$$

Our Cayley graph for this example looks like the following:



Note: We are writing our group elements as products rather than ordered triples. So $(x, y, z) = xyz$.

Also note that our Cayley graph is 3-regular, that is, every vertex in $Cay(S;\ G)$ has degree 3. This is because $|S| = 3$ and the edges of the graph arise from multiplication by elements of S.

1.3 Distance Regular Graphs

Now suppose we have a (connected, simple) graph, call it H.

Given two vertices x and y in H of distance i apart, define:

$$P^i_{j,k}(x,y) = |\{z|\ d(x,z) = j, d(y,z) = k\}|$$

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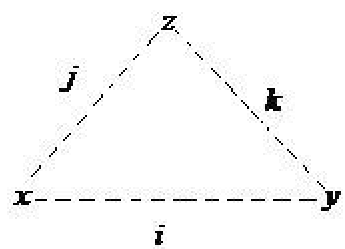


We choose a pair of vertices which are distance i apart. Next, we look at all the vertices which are distance j from x and also distance k from y. The number of vertices that satisfy this is exactly our $P^i_{j,k}(x,y)$.

Definition 3: A graph is *distance regular* if $P^i_{j,k}(x,y)$ is independent of x and y and only depends on i , j and k.

This means that, for any pair of vertices in a distance regular graph, the value for $P^i_{j,k}(x,y)$ should be a constant value.

Definition 4: A graph is *strongly regular* if it is distance regular and has diameter 2.

To have a nicer way to check for distance-regularity, we define the following parameters:

$$c_i = P^i_{i-1,1}$$
$$a_i = P^i_{i,1}$$
$$b_i = P^i_{i+1,1}$$

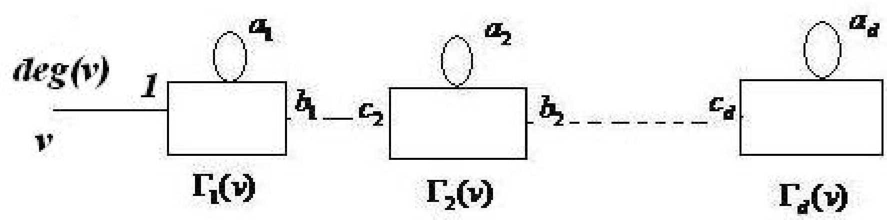
Notice that, in each of these parameters, the pair of vertices (x, y) is distance i apart and the vertices we are counting (z) are adjacent to y. The only difference is the distance from x to z, which varies from $i-1$ to i to $i+1$.

Note: For *strongly regular graphs* we define:

$$\lambda = P^1_{1,1}$$
$$\mu = P^2_{1,1}$$

The definitions of the previous parameters might suggest that we look at the neighborhoods of an arbitrary vertex, call it v.

Now using these parameters, we create something called an *adjacency diagram*:



Informally, we are looking at traveling from neighborhood to neighborhood. Assume we have picked a vertex, call it w, out of an arbitrary neighborhood, $\Gamma_i(v)$. The value c_i is the number of vertices in the $(i-1)$st neighborhood which are adjacent to w. The value a_i is the number of vertices in the same neighborhood of w which are adjacent to w. Finally, the value for b_1 is the number of vertices in the $(i+1)$st neighborhood which are adjacent to w.

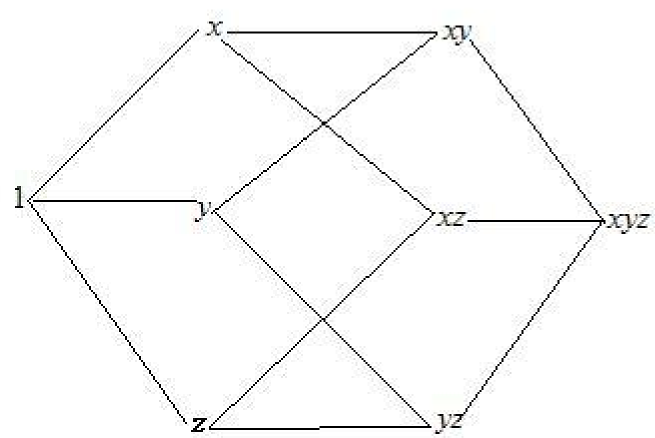
Note: d is the diameter of our graph.

Aside: Since our graph can be assumed to be regular (all vertices have the same degree) because a Cayley graph is regular, then we get that:

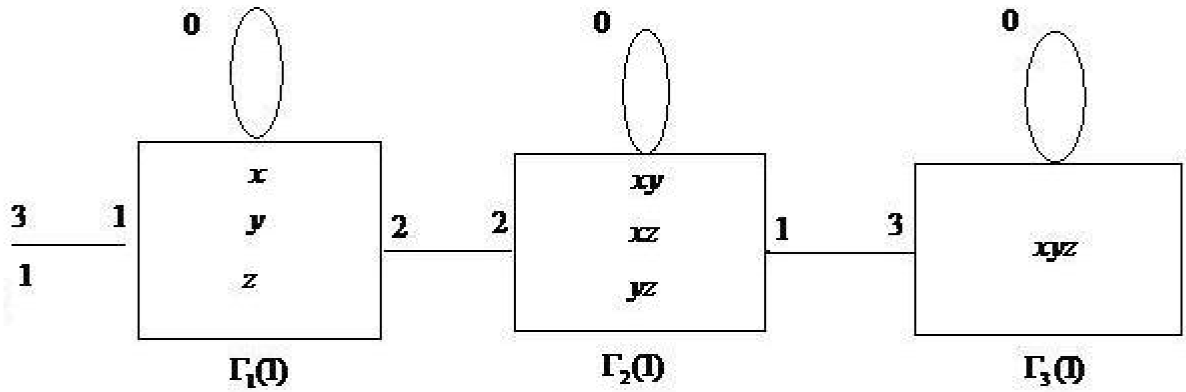
$$a_j + b_j + c_j = deg(v)$$

An Example of a Distance Regular Graph

Below is the Cayley graph from the previous example.



We will now create the adjacency diagram for this example.



So since all our $a_i's$, $b_i's$ and $c_i's$ are independent of the vertex v we choose and also constant for each vertex in a given neighborhood, then our *Cayley graph* is also *distance regular*. Hence, we have a *distance regular Cayley graph*.

2 Background and Theory

We will now introduce the relevant theory used throughout the project. First, we will define characters. Next, we will define idempotents. We will, then, explain the relationship between characters and idempotents and explain the roles they will play in the study of distance regular Cayley graphs.

2.1 Characters

Definition 5: A character, χ, is a homomorphism from a group, G, into the complex numbers $\mathbb{C}$. That is, $\chi : G \longrightarrow \mathbb{C}$.

Given a group G, we will look at the *group ring*

$$\mathbb{C}\ [\mathrm{G}]=\{\textstyle\sum_{g\in G}\ \alpha_g g \mid \alpha_g \in \mathbb{C}\ \}$$

as a vector space over $\mathbb{C}$ with basis G.

G is the standard basis for $\mathbb{C}[G]$. However, during this project, we will do a change of basis to a "better" basis called the *Fourier basis.*

2.2 Idempotents

Definition 6: Given a character χ: $G \longrightarrow \mathbb{C}$, define:

$$e_\chi = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\cdot g$$

where $\overline{\chi(g)}$ acts as a coefficient in $\mathbb{C}$ for the group ring.

Fact: The $e_\chi{}'s$ have the property that:

$$e_{\chi_1}e_{\chi_2} = \begin{cases} e_{\chi_1} & \text{if; } \chi_1 = \chi_2 \\ 0 & \text{if } \chi_1 \neq \chi_2 \end{cases}$$

So the e_χ's have the property of *orthogonality*.

Also, we have that $e_\chi^2 = e_\chi$. This is exactly the the definition of an *idempotent*.

This means that the seemingly arbitrary definition of e_χ as a sum of elements in the group ring is actually the precise definition of an idempotent in the group ring.

2.3 Characters and Idempotents

Let $G^\star$ be the set of all the characters of our group G.

Fact: If G is an abelian group, then $\{e_\chi | \chi \in G^\star\}$ forms a basis for the group ring $\mathbb{C}$ [G].

This is the basis we will be using throughout the rest of the project.

However, we will soon adopt the notation of writing our idempotents as matrices rather than linear combinations of group elements.

This is a very important step since we are translating from the arduous task of manipulating linear combinations of group ring elements to the much easier task of manipulating matrices. We have essentially shifted from algebraic techniques to linearly algebraic techniques.

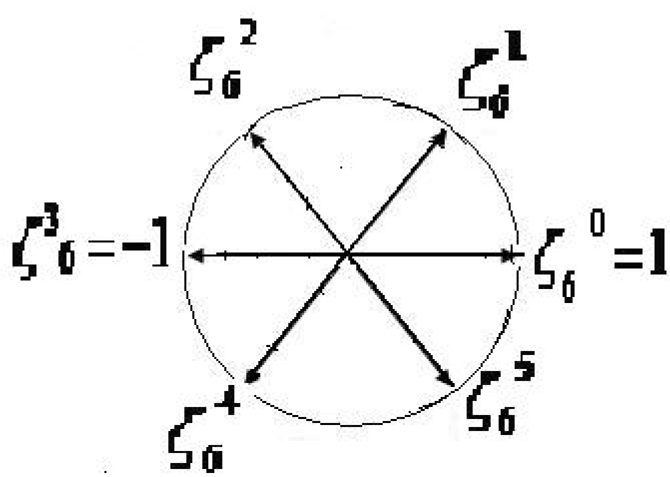
Now let us look at an example...

An Example of an Idempotent in a Group Ring

Let $G = C_6 = \{x|\ x^6 = 1\}$ be our group.

Then we have a total of 6 characters since the order of the group is 6.

In our case, χ_i: $x \longrightarrow \zeta_6^i$ for $0 \leq i \leq 5$.



Let's just look at χ_1: $x \longrightarrow \zeta_6^1$.

Then from our original definition of an idepotent as a sum of group elements with complex coefficients, we get that:

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot g$$

So in this example we get:

$$e_{\chi_1} = \frac{1}{|C_6|} \sum_{g \in C_6} \overline{\chi(g)} \cdot g$$

$$= \tfrac{1}{6}(\overline{\chi(1)}{\cdot}1 + \overline{\chi(x)}{\cdot}x + \overline{\chi(x^2)}{\cdot}x^2 + \overline{\chi(x^3)}{\cdot}x^3)\ \overline{\chi(x^4)}{\cdot}x^4 + \overline{\chi(x^5)}{\cdot}x^5)$$

$=\frac{1}{6}(1\cdot 1\ +\ \overline{\zeta_6}\cdot x\ +\ \overline{\zeta_6^2}\cdot x^2\ +\ \overline{\zeta_6^3}\cdot x^3\ +\ \overline{\zeta_6^4}\cdot x^4\ +\ \overline{\zeta_6^5}\cdot x^5)$

$=\frac{1}{6}(1\cdot 1\ +\zeta_6^5\cdot x\ +\ \zeta_6^4\cdot x^2 - x^3\ +\ \zeta_6^2\cdot x^4\ +\ \zeta_6\cdot x^5)$

$=\frac{1}{6}[1,\ \zeta_6^5,\ \zeta_6^4,\ -1,\ \zeta_6^2,\ \zeta_6]$

The last line exhibits the transition from a sum to a matrix array.

Note: The matrix array above consists of the coefficients of our group elements. This is because the location of the entries in the matrix implies which group element each coefficient corresponds to.

To explicitly match coefficients with group elements, we can create a "template" which consists of the group elements in their respective positions in the array.

The template for the previous example is given below:

$$[1, x, x^2, x^3, x^4, x^5]$$

2.4 Rational Idempotents

For our purposes, however, we want to deal with *rational idempotents*, which are simply idempotents with rational entries. Right now, though, our idempotents have complex entries.

Fact: There is a function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ called a *field (Galois) automorphism* which essentially fixes the rational entries and conjugates the complex entries of our idempotent matrix.

We also define an equivalence relation on the characters by:

$$\chi \sim \chi'\ if\ Ker(\chi) = Ker(\chi')$$

So we consider two characters to be equivalent if they have the same kernel; that is, if they send the same group elements to the identity.

We now sum all idempotents which correspond to characters of the same kernel.

Define:

$$[e_\chi] = \sum_{\chi \sim \chi'} e_{x'}$$

Then since we are essentially summing rational entries with rational entries and complex entries with their conjugates, then the matrix sum should be a matrix with rational entries.

Hence, the $[e_\chi]'s$ are exactly our desired *rational idempotents*.

3 Important Fact

Now with enough background in the necessary definitions and theory, we will present a very important fact which allows us to bridge between distance regular graphs and Cayley graphs to look for distance regular Cayley graphs.

3.1 Eigenvalues of a Distance Regular Graph

Since *distance regular graphs* are a type of graph, we can create something called an *adjacency matrix* for the graph which gives information about how the vertices of the graph are connected. The adjacency matrix is a matrix whose entries are either 0 or 1. The entry is 0 if the pair of vertices corresponding to that entry are nonadjacent and the entry is 1 if the pair of vertices corresponding to that entry are adjacent. From this adjacency matrix we can use linear algebra to find its *eigenvalues* and multiplicities.

3.2 Character Values of a Cayley Graph

Since *Cayley graphs* come from groups, we can apply the characters of the group G to the set S to get the verious *character values* of S and the multiplicities of these character values.

3.3 Important Fact

Given a *distance regular Cayley graph*, the *eigenvalue*s of the adjacency matrix and their multiplicities correspond exactly to the *character values* of S and their multiplicities.

4 Strongly Regular Graphs

We began by first looking at strongly regular graphs, which are distance regular graphs of diameter two.

4.1 Explanation of Parameters

We began our project by looking at strongly regular graphs and trying to see if any of these were also Cayley graphs. Since extensive work has already been done on strongly regular graphs, we were able to obtain tables of information on existing strongly regular graphs.

Specifically, we were given parameter sets (v, k, λ, μ) where v is the number of vertices in the graph, k is the degree of the graph, λ is the number of common neighbors for an adjacent pair of vertices in the graph, and μ is the number of common neighbors for a nonadjacent pair of vertices in the graph.

Using definitions from earlier, recall that we can also write λ and μ as:

$$\lambda = P^1_{1,1}$$

$$\mu = P^2_{1,1}$$

Additionally, we were given the eigenvalues and multiplicities of the graph, which are really the eigenvalues and multiplicities of the adjacency matrix of the graph.

4.2 First Project Goal

So since we already have all of the information we need for a strongly regular graph, i.e. $(v,\ k,\ \lambda,\ \mu)$ and the eigenvalues and multiplicities, then the goal of this part of the project was to try to build the set S from the rational idempotents of a given group, G.

So let G be a finite abelian group of order $|G|$.

Fact: Given a group G, the number of characters of G is exactly the order of G.

We want to use the *characters* of G to form the *rational idempotents* for our group G. Since the idempotents form the basis elements of our group ring, then the rational idempotents are a sum of a given number of basis elements of the group ring.

Our goal is to form the subset S as a sum of rational idempotents.

Note: Since we have switched the a matrix array notation, then our original group G would be represented as simply a ones matrix where the number of entries is equal to the order of G.

So since S is a subset of G, then either S contains a group element or it does not. Hence, S needs to be a matrix whose entries are only 0 and 1.

Then we have that the set S is an element of the group ring $\mathbb{C}$ [G]. But now our basis elements are idempotents. So we get:

$$S = \sum_{\chi} \alpha_\chi \cdot e_\chi$$

Aside: Let us apply a character, ϕ, to an idempotent e_χ. We will get two cases, the first case being when $\phi = \chi$ and the second case being when $\phi \neq \chi$.

Case 1: Suppose $\phi = \chi$. Then:

$$\chi(e_\chi) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot \chi(g) = \frac{1}{|G|} \sum_{g \in G} 1 = \frac{1}{|G|} \cdot |G| = 1$$

Case 2: Suppose $\phi \neq \chi$. Then we get:

$$\phi(e_\chi) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot \phi(g) = 0$$

Note: We get 0 by a property of characters.

So we get that a character times an idempotent is 1 of the character and the character corresponding to the idempotent are the same and we get 0 otherwise.

Using this fact, let now apply a character, ϕ to our set S. Then we get:

$$\phi(S) = \sum_{\chi'} \alpha_\chi \cdot \phi(e_\chi) = \alpha_\chi \cdot \chi(e_\chi) = \alpha_\chi$$

by using the previous result. So, by definitions of S, **the coefficients of the idempotents are the character values of** S.

Now we rely heavily on the **Important Fact**...

Recall that the fact tells us that, given a distance regular Cayley graph, the character values of S and their multiplicities correspond exactly to the eigenvalues and multiplicities.

So now once we form our rational idempotents by looking at the characters of our group G, we can now use the eigenvalues and multiplicities of the distance regular graphs given to us to see is we really can form the set S.

If we actually DO form a set S consisting of only 0's and 1's, then we have a *strongly regular Cayley graph.*

4.3 Finding a Strongly Regular Cayley Graph

To get a better idea of how to form rational idempotents and how to apply all of the theory previously introduced. we will provide a very useful example.

Let us look at a specific parameter set for a strongly regular graph:

$$(v, k, \lambda, \mu) \;=\; (36, 14, 4, 6)$$

eigenvalues: 14, 2, -4
multiplicities: 1, 21, 14

This is all of the information needed to begin forming the rational idempotents of a given group. But first we need to choose a group.

Let out group be:

$$G = C_6 \times C_6 = \{x, y : x^6 = y^6 = 1, xy = yx\}$$

Then our 6×6 idempotent matrices will have entries which are the coefficients of the following group elements:

$$\begin{bmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 \\ y & xy & x^2y & x^3y & x^4y & x^5y \\ y^2 & xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & x^5y^2 \\ y^3 & xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & x^5y^3 \\ y^4 & xy^4 & x^2y^4 & x^3y^4 & x^4y^4 & x^5y^4 \\ y^5 & xy^5 & x^2y^5 & x^3y^5 & x^4y^5 & x^5y^5 \end{bmatrix}$$

Since the order of our group is 36, then there are 36 characters total.

By definition, each character sends x and y to a power of a sixth root of unity.

$$\chi_{i,\,j} : \begin{array}{l} x \longrightarrow \zeta_6^i \quad \text{for} \quad 0 \le i \le 5 \\ y \longrightarrow \zeta_6^j \quad \text{for} \quad 0 \le j \le 5 \end{array}$$

Let us look at two of the characters of our group to see how they work and to see how to form a rational idempotent.

Ex:

$\chi_{1,1} : x \longmapsto \zeta_6 \; y \longmapsto \zeta_6$

$\chi_{5,5} : x \longmapsto \zeta_6^5 \; y \longmapsto \zeta_6^5$

Note: $\chi_{1,1}$: $xy^5 \longmapsto \zeta_6 \zeta_6^5 = 1$

$\chi_{5,5}$: $xy^5 \longmapsto \zeta_6^5 (\zeta_6^5)^5 = \zeta_6^{30} = 1$

So the kernels (the group elements which get mapped to the identity 1) for the above characters are:

$Ker(\chi_{1,1}) =< xy^5 >$

$Ker(\chi_{5,5}) =< xy^5 >$

So these two characters have the same kernel and actually there are the only two characters with this kernel. So by how we originally defined the equivalence relation between the characters of a group, we have that these characters are equivalent.

So since our rational idempotents have been defined as a sum of the idempotents which correspond to equivalent characters, then we need to sum the two idempotent matrices which correspond to our two characters $\chi_{1,1}$ and $\chi_{5,5}$.

The corresponding idempotents are:

$$e_{1,1} = \frac{1}{36} \begin{bmatrix} 1 & \zeta_6^5 & \zeta_6^4 & -1 & \zeta_6^2 & \zeta_6 \\ \zeta_6^5 & \zeta_6^4 & -1 & \zeta_6^2 & \zeta_6 & 1 \\ \zeta_6^4 & -1 & \zeta_6^2 & \zeta_6 & 1 & \zeta_6^5 \\ -1 & \zeta_6^2 & \zeta_6 & 1 & \zeta_6^5 & \zeta_6^4 \\ \zeta_6^2 & \zeta_6 & 1 & \zeta_6^5 & \zeta_6^4 & -1 \\ \zeta_6 & 1 & \zeta_6^5 & \zeta_6^4 & -1 & \zeta_6^2 \end{bmatrix}$$

$$e_{5,5} = \frac{1}{36}\begin{bmatrix} 1 & \zeta_6 & \zeta_6^2 & -1 & \zeta_6^4 & \zeta_6^5 \\ \zeta_6 & \zeta_6^2 & -1 & \zeta_6^4 & \zeta_6^5 & 1 \\ \zeta_6^2 & -1 & \zeta_6^4 & \zeta_6^5 & 1 & \zeta_6 \\ -1 & \zeta_6^4 & \zeta_6^5 & 1 & \zeta_6 & \zeta_6^2 \\ \zeta_6^4 & \zeta_6^5 & 1 & \zeta_6 & \zeta_6^2 & -1 \\ \zeta_6^5 & 1 & \zeta_6 & \zeta_6^2 & -1 & \zeta_6^4 \end{bmatrix}$$

Then:

$$[e_{1,1}] = e_{1,1} + e_{5,5} = \frac{1}{36}\begin{bmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ -1 & -2 & -1 & 1 & 2 & 1 \\ -2 & -1 & 1 & 2 & 1 & -1 \\ -1 & 1 & 2 & 1 & -1 & -2 \\ 1 & 2 & 1 & -1 & -2 & -1 \end{bmatrix}.$$

After looking at 36 characters of $C_6 \times C_6$ we found that there were 20 rational idempotents and for the following multiplicities (1,21,14) there were nearly 125 ways of matching our eigenvalues (14, 2,-4) with our rational idempotents.

Important Note: When we are summing the rational idempotents, we must clearly multiply our rational idempotents only by a single eigenvalue. Hence, if a rational idempotent is the sum of k idempotents, then the eigenvalue MUST have multiplicity at least k.

In this specific case, there are nearly 125 ways of matching idempotents and eigenvalues. Since there are so many possibilities, we will look at the *homomorphic images* of $C_6 \times C_6$ to form the set S.

4.4 Homomorphic Images

What we will be doing is applying various homomorphisms to the group which will send various group elements to the identity.

Because of this, some of the 20 rational idempotents will be sent to the zero matrix. So when we look for the image of S, there will be fewer possibilities for which eigenvalues can be the coefficients for the rational idempotents.

We can, then, use the information we get from the images of S to find our set S in the original group.

These are the homomorphisms which gave our images:

f_1: $C_6 \times C_6 \longrightarrow C_6 \times C_1$
$x \longmapsto x$, $y \longmapsto 1$

If we look at how we formed our 6×6 matrix, it can be seen that applying the homomorphism corresponds to adding up all 6 rows in our 6×6 matrix.

f_2: $C_6 \times C_6 \longrightarrow C_6 \times C_2$
$x \longmapsto x$, $y^2 \longmapsto 1$

In this case, applying the homomorphism corresponds to adding up every third row in our 6×6 matrix.

f_3: $C_6 \times C_6 \longrightarrow C_6 \times C_3$
$x \longmapsto x$, $y^3 \longmapsto 1$

In this last case, applying the homomorphism corresponds to adding up every other row in our 6 $\times 6 matrix$.

Note: We will reject all images which contain any entries which are not 0 or 1, that is, negative entries, fraction entries, and entries of 2 or larger.

So by appylyling each of the three homomorphisms to our group $C_6 \times C_6$ we get the images of the original 20 idempotents. Since some of these will go to the zero matrix, we will have fewer rational idempotents to deal with when we are looking at the possibilities of eigenvalue coefficients. Hence, we should be able to, with some work, find the possible images of our set S under each of the homomorphisms.

We can then create tables which list the image matrix, the rational idempotents with their corresponding eigenvalue coefficients, and also information about how many idempotents are being summed together to make each rational idempotent.

4.5 Using Images to Construct the Set S

Looking at our table of images, we begin first with an image of S in $C_6 \times C_3$, seen as the first matrix below. Then, since the homomorphism that gives that image is equivalent to adding every other row in our 6×6 matrix S, then entries of 2 in the $C_6 \times C_3$ image must come from the addition of two ones in the final matrix S. Similarly, entries of 0 in the $C_6 \times C_3$ image must come from the addition of two zeros in the final matrix S. This can be seen in the second matrix below.

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 1/2 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 1/2 & 1 & 1/2 \end{bmatrix}$$

Note: The entries of $\frac{1}{2}$ above can be though of as representing a probability of $\frac{1}{2}$ of that entry having either an entry of 0 or 1.

Next, we apply the homomorphism f_3 only with x and y switched. So now we add up every other column in our 6×6 matrix. Then, since we can apply various *automorphisms* to our set S, then the $C_3 \times C_6$ image which results should correspond to an existing $C_6 \times C_3$ image. However, we cannot directly add the entries which have entries of 1 since we do not know if they are a 0 or a 1. So we only have partial information about our images. But looking through our table we find that there is an image which satisfies the conditions required. This image is the third matrix below.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 1/2 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 1/2 & 1 & 1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ & & 1 \\ & 1 & \\ 1 & 0 & 0 \\ & & 1 \\ & 1 & \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Now with our $C_3 \times C_6$ image, we get more information about our desired matrix S from the new entries of 0 or 2. This new information is shown in the green entries of the second matrix below. We have represented the other entries by variables a, $\bar{a}$, b, $\bar{b}$, c, $\bar{c}$, d, $\bar{d}$. These represent either entries of 0 or 1 and we have that $a+\bar{a}=1$ and similarly for b, c, and d.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \mathsf{a} & 1 & 1 & \mathsf{c} & 1 & 0 \\ \mathsf{b} & 0 & 0 & \mathsf{d} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \bar{\mathsf{a}} & 0 & 1 & \bar{\mathsf{c}} & 0 & 0 \\ \bar{\mathsf{b}} & 0 & 1 & \bar{\mathsf{d}} & 1 & 1 \end{bmatrix}$$

Now we apply a different homomorphism, f_2 to the first matrix below. As done before, we look for images in our table which have the same non-variable entries as the second matrix below. From that such image in $C_6 \times C_2$ we can solve for all of our variables. This also allows us to complete the formation of our set S, with the new entries in red. As a check, we again apply the homomorphism f_2, only with x and y switched again. Indeed, this again corresponds to the transpose of one of our $C_6 \times C_2$ images.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \mathsf{a} & 1 & 1 & \mathsf{c} & 1 & 0 \\ \mathsf{b} & 0 & 0 & \mathsf{d} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \bar{\mathsf{a}} & 0 & 1 & \bar{\mathsf{c}} & 0 & 0 \\ \bar{\mathsf{b}} & 0 & 1 & \bar{\mathsf{d}} & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathsf{b}+\bar{\mathsf{a}} & 0 & 1 & 1+\mathsf{d}+\bar{\mathsf{c}} & 1 & 0 \\ \mathsf{a}+\bar{\mathsf{b}} & 1 & 2 & \mathsf{c}+1+\bar{\mathsf{d}} & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathsf{b}+\bar{\mathsf{a}} & 0 & 1 & 1+\mathsf{d}+\bar{\mathsf{c}} & 1 & 0 \\ \mathsf{a}+\bar{\mathsf{b}} & 1 & 2 & \mathsf{c}+1+\bar{\mathsf{d}} & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 3 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}.$$

So translating back to the notation of writing linear combinations of group elements, we get the following for S:

$S = x^3 + y + xy + x^2y + x^4y + x^3y^2 + x^4y^2 + x^3y^3 + x^2y^4 + x^3y^4 + y^5 + x^2y^5 + x^4y^5 + x^5y^5$

As one check we can apply each of the 36 characters to S. If we get that the character values of S and their multiplicities correspond exactly to the eigenvalues and multiplicities we began with, then we do indeed have a strongly regular Cayley graph.

Note: This method is not unique to diameter 2. We can apply a similar technique later for graphs of diameter 3.

Our next section deals with looking for the *non*existence of distance regular Cayley graphs.

5 Nonexistence of Distance Regular Cayley Graphs

We will look at some methods for proving the nonexistence of a distance regular Cayley graph.

5.1 Nonexistence (Example) Type 1

Groups	Parameters (v, k, λ, μ)	θ	m_θ	τ	m_τ
$C_7 \times C_7$	(49,16,3,6)	2	32	-5	16

Reason: When summing idempotents to get rational idempotents, we get that, for the nontrivial case, 6 idempotents must be added together to yield a rational idempotent. But the multiplicities of the eigenvalues are NOT multiples of 6. So this violates the fact that a rational idempotent can only have a single eigenvalue as a multiplier.

So we need to note exactly how many idempotents are being added up to make our rational idempotents.

This is because these equivalent idempotents are coming from equivalent characters. So if our rational idempotent is the sum of m equivalent idempotents, then this rational idempotent needs to be multiplied by an eigenvalue of a distance regular graph with multiplicity at least m. However, this must hold true as we continue to pair eigenvalues with rational idempotents. Hence, while this requirement may not be violated originally, it may be violated when trying to evenly distribute the eigenvalues evenly.

5.2 Nonexistence Type 2

The next type of nonexistence argument can be made when the eignevalues DO distribute evenly amongst the rational idempotents.

After trying every possibility of matching eigenvalues with rational idempotents, we may get that every possible image S contains entries other than 0 or 1. If this occurs, then we have successfully ruled out the existence of a distance regular Cayley graph.

The next page includes some graphs which are NOT distance regular Cayley graphs.

Some Strongly Regular Graphs that are not Cayley Graphs

Groups	Parameters (v, k, λ, μ)	θ	m_θ	τ	m_τ	Reasons
C_9	(9,4,1,2)	1	4	2	4	Type 1
$C_{10} \cong C_5 \times C_2$	(10,3,0,1)	1	5	-2	4	Type 2
$C_{15} \cong C_5 \times C_3$	(15,6,1,3)	1	9	-3	5	Type 1
$C_{21} \cong C_7 \times C_3$	(21,10,3,6)	1	14	-4	6	Type 2
$C_{21} \cong C_7 \times C_3$	(21,10,5,4)	3	6	-2	14	Type 2
$C_7 \times C_7$	(49,16,3,6)	2	32	-5	16	Type 1

Some diameter 3 Distance Regular Graphs that are not Cayley Graphs

v	Groups	Parameters $[b_0, b_1, b_2][c_1, c_2, c_3]$	Eigenvalues Multiplicities	Reason
15	$C_5 \times C_3$	[4,2,1] [1,1,4]	[4,2,-1,-2] [1,5,4,5]	Type 1
16	C_{16}	$[7, 2, 1][1, 2, 7]$	$[7, -1, (1/2)\{\pm 2 \pm \sqrt{32}\}]$ [1,7,4,4]	Type 1
16	$C_8 \times C_2$	$[7, 2, 1][1, 2, 7]$	$[7, -1, (1/2)\{\pm 2 \pm \sqrt{32}\}]$ [1,7,4,4]	Type 2
20	$C_{20} \cong C_5 \times C_4$	[9,4,1] [1,4,9]	[9,3,-1,-3] [1,5,9,5]	Type 1
	$C_5 \times C_2 \times C_2$	[9,4,1] [1,4,9]	[9,3,-1,-3] [1,5,9,5]	Type 23
24	$C_{24} \cong C_8 \times C_3$	[8,5,2] [1,4,4]	[8,4,0, -4] [1,3,15,5]	Type 1
	$C_{12} \times C_2$	[8,5,2] [1,4,4]	[8,4,0, -4] [1,3,15,5]	Type 2
24	$C_{24} \cong C_8 \times C_3$	[11,1,1][1,1,11]	[11,-1,$(1/2)\{\pm 8 \pm \sqrt{108}\}$] [1,11,6,6]	Type 12
	$C_{12} \times C_2$	[11,1,1][1,1,11]	$[11, -1, (1/2)\{\pm 8 \pm \sqrt{108}\}]$ [1,11,6,6]	Type 12

6 Construction Methods

Before introducing the construction methods, a very important lemma will be introduced. We will then look at two methods for actually *constructing* distance regular Cayley graphs. The first method requires a specific group and, using nice subgroups of this group, gives a specific form for the subset S. Using the lemma, we show that the given Cayley graph is also distance regular. Next, we use a specific type of distance regular graph of given diameter and take the union of the first and last neighborhoods to yield a distance regular graph of smaller diameter.

6.1 Important Lemma

The following lemma states that a character value of a subgroup is either 0 or the order of that subgroup.

Lemma: Given a finite abelian group G and a subgroup H of G we have that, given a character χ of G, either $\chi(H) = 0$ or $\chi(H) = |H|$.

From this lemma we get two corollaries which generate specific types of distance regular Cayley graphs.

Corollary: Given a Cayley graph with $G = C_n \times C_n$ and $S =< x > + < y > -2$, the character values for S [and multiplicities] are given by:

$$\chi_{i,j}(s) = \begin{cases} 2n-2 \text{ if } i=j=0 \text{ [once]} \\ n-2 \text{ if } i=0, j\neq 0 \text{ or } i\neq 0, j=0 \text{ [}2n-2\text{ times]} \\ -2 \text{ otherwise [}(n-1)^2\text{ times]} \end{cases}$$

These will correspond exactly with the eigenvalues and multiplicities of a strongly regular graph.

Corollary: Given a Cayley graph with $G = C_n \times C_n \times C_n$ and $S =< x > + < y > + < z > -3$, the character values for S are given by:

$$\chi_{i,j}(s) = \begin{cases} 3n-3 \text{ if } i=j=k=0 \text{ [once]} \\ 2n-3 \text{ if } i=j=0, k\neq 0 \text{ or } j=k=0, i\neq 0 \text{ or } i=k=0, j\neq 0 \text{ [}3n-3\text{ times]} \\ n-3 \text{ if } i,j\neq 0, k=0 \text{ or } j,k\neq 0, i=0 \text{ or } i,k\neq 0, j=0 \text{ [}3(n-1)^2\text{ times]} \\ -3 \text{ otherwise [}(n-1)^2\text{ times]} \end{cases}$$

These will correspond exactly with the eigenvalues and multiplicities of a distance regular graph of diameter 3.

Note: We can continue to generalize by using similar groups G and subsets S. This will generalize to a type of graph called a *Hamming graph*.

6.2 Fusion Method

In addition to looking at subgroups, we have also looked at another construction method which creates new distance regular graphs from known distance regular graphs.

We have been calling this method *fusion* because we are constructing the set S by taking the union of two neighborhoods in an existing distance regular Cayley graph and trying to create a new distance regular Cayley graph of smaller diameter.

An Example of Fusion

Here is an instructive example of the method:

Let our group be:

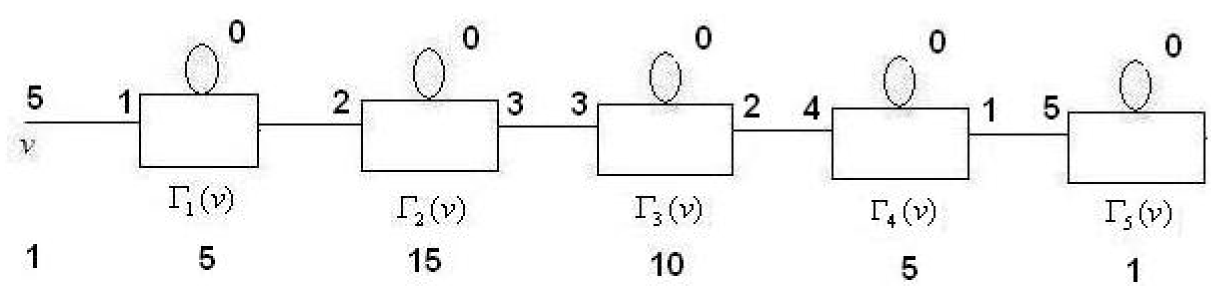
$$G = C_2 \times C_2 \times C_2 \times C_2 \times C_2$$

and our set S be:

$$S = x_1 + x_2 + x_3 + x_4 + x_5$$

This graph is actually our *Hamming graph*. So from a generalization of the previous corollaries, we have that the graph is distance regular of diameter 5.

Below is the adjacency diagram for this graph:



Note: Below the diagram is the number of vertices in each neighborhood.

Now fuse the 1st and 5th neighborhoods by making the new 1st neighborhood the old 1st neighborhood union the 5th neighborhood.

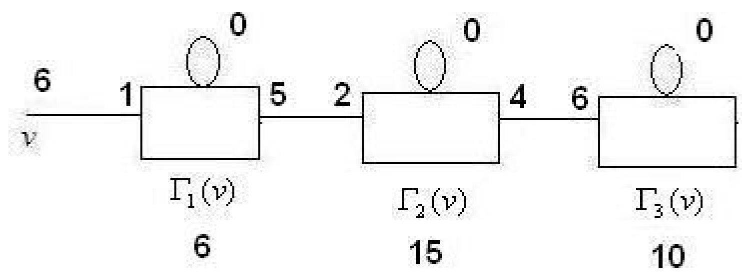
So now our group is:

$$G = C_2 \times C_2 \times C_2 \times C_2 \times C_2$$

ans our new set S is:

$$S = x_1 + x_2 + x_3 + x_4 + x_5 + x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5$$

Next, construct the Cayley graph with the same group and the new set S. From this new Cayley graph, we get the following adjacency diagram:



This corresponds exactly to the parameters for an existing distance regular graph...so we now have constructed a **distance regular Cayley graph** of diameter 3 from a distance regular graph of diameter 5!

So this method may be able to be extended to different direct products of groups to yield distance regular Cayley graphs of various diameters...

7 Conclusion

This project has produced both existence and nonexistence results of distance regular Cayley graphs. First, a method for finding distance regular Cayley graphs was found by using the Fourier basis of the group ring $\mathbb{C}$ [G] rather than the standard basis. This way, we were able to, by representing the idempotents of a group G as matrix arrays, manipulate linear algebra rather than abstract algebra.

Moreover, the use of rational idempotents also creates some simple ways to rule out the existence of distance regular Cayley graphs. So we now have an efficient way to look for the existence as well as the nonexistence of distance regular Cayley graphs.

Finally, we also looked at two different method of trying to construct families of distance regular Cayley graphs. The first method takes advantage of a very important lemma and uses nice subgroups of our group to form a specific type of set S. Then, by looking in specific type of groups, we are able to find an infinite family of distance regular graphs of all diameters. This method also exploits some nice properties of *Hamming graphs*.

The second method also uses *Hamming graphs*. By beginning with an almost trivial type of Hamming graph, we are able to fuse the first and last neighborhoods of an existing distance regular graph to find distance regular graphs of smaller diameter.

While a number of the existence results are either already known or somewhat trivial to find, the idempotent method of searching for distance regular graphs seems to be a promising method. Also, some nice nonexistence arguments can be made by manipulating the same method. Finally, the second method of construction, which we called *fusion*, could possibly lend itself to a techniques for using existing distance regular Cayley graphs to find NEW distance regular Cayley graphs.