On the minimum semidefinite rank of a graph using vertex sums, graphs with $msr(G) = |G| - 2$, and the msrs of certain graph classes.[*†]

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Abstract

Given a Hermitian matrix $A \in M_n(\mathbb{C})$, associate a simple, undirected graph $G(A)$ where $V(G) = \{1, 2, \ldots, n\}$ and $E(G) = \{\ ij \ \mid \ a_{ij} \neq 0, i \neq j\ \}$. The collection of all Hermitian matrices that share a common graph G is denoted $\mathcal{H}(G)$. The problem of finding the multiplicities of the eigenvalues among the matrices in $\mathcal{H}(G)$ has received much attention recently. In this report, we consider $\mathcal{P}(G) \subset \mathcal{H}(G)$ where $\mathcal{P}(G)$ is the set of all positive semidefinite matrices corresponding to G. The *minimum semidefinite rank* of G, denoted $msr(G)$, is defined to be the minimum rank among all matrices in $\mathcal{P}(G)$.

In this paper we present results on finding the $msr(G)$ when G is written as a vertex sum of two graphs G_1 and G_2 that share a cut set of at most two vertices. A classification of all graphs with $msr(G) = |G| - 2$ is given. An upper bound for $msr(G)$ when G is obtained from G_1 and G_2 by cancellation of edges in a star forest is also presented. Moreover, the msr of cartesian product, strong product, corona, and line graphs of certain graphs are proven.

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1 Introduction

A combinatorially symmetric matrix, A, naturally corresponds to a simple, undirected graph $G(A)$ with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{ij \mid a_{ij} \neq 0, i \geq j\}$. The graph is independent of the diagonal entries of the matrix. Particular examples, such as the Laplacian matrix of a graph, have shown correspondences between graph and matrix properties [6].

This relationship between matrices and graphs has been used to study the structure of multiplicities of eigenvalues of Hermitian matrices [11]. For the class of all complex Hermitian matrices, determining possible eigenvalue multiplicities, or even finding the highest possible multiplicity among all eigenvalues seems to be an intractible problem. In this paper, as in [3], the related problem of the minimum rank among positive semidefinite (PSD) matrices with a given graph is considered:

Given a graph G, let $\mathcal{P}(G) = \{A \text{ is PSD} \mid G(A) = G\}$. Define the *minimum semidefinite rank* of G as

$$msr(G) = \min\{rank(A) : A \in \mathcal{P}(G)\}.$$

Furthermore, if G is not connected then any $A \in \mathcal{P}(G)$ is a direct sum of PSD matrices corresponding to the connected components of G. Since $rank(A)$ is the sum of the ranks of its diagonal blocks we assume without loss of generality that G is a connected graph. One upper bound can easily be shown for the msr. The Laplacian matrix of a graph G on n vertices, denoted $\Delta(G)$ is a positive semidefinite matrix and has rank $n - 1$ [6]. Therefore $msr(G) \leq |G| - 1$ for all graphs G. More importantly, this proves that $\mathcal{P}(G)$ is nonempty for any graph G.

1.1 Graph Theory Preliminaries

The following notation will be used throughout this report. For more graph theory background see [8]. For any graph G, the order of G, denoted $|G|$, is the number of vertices in G. We say two vertices u and v are adjacent if $uv \in E(G)$. If $v \in V(G)$ we define the *open neighborhood* of v, denoted $N(v)$, to be the set of all vertices in $V(G)$ adjacent to v. The *closed neighborhood* of v, denoted $N[v]$, is the set $N(v) \cup \{v\}$. If $u, v \in V(G)$ and $N[u] = N[v]$ then u and v are *duplicate vertices.*

The complete graph on n vertices, K_n, is a graph where every pair of vertices is adjacent. As a matrix with every entry one is PSD and has rank one it is clear that $msr(K_n) = 1$. The cycle on n vertices, denoted C_n, is $C_n = (V, G)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_nv_1\}$. It is well known that $msr(C_n) = n - 2$ [10]. Let P_n denote the *path* on n vertices, that is P_n is a cycle on n vertices with an edge missing; then $msr(P_n) = n - 1$ [10]. If G is a connected graph without any cycles then G is a *tree.* It is a well known result that G is a tree if and only if $msr(G) = |G| - 1$ [10]. A *forest of trees* is a graph composed of some number of disjoint trees. A star is a connected graph with no more than one vertex having a degree greater than one. A star forest is graph composed of some number of disjoint stars. Any graph with 0, 1, or 2 edges is a star forest.

A graph $G' = (V', E')$ is a *subgraph* of graph $G = (V, E)$ if $V' \subseteq V, E' \subseteq E$. The subgraph $G[S]$ of $G = (V, E)$ *induced* by $S \subseteq V$ is the subgraph with vertex set S and edge set $\{ij \in E \mid i, j \in S\}$. If H is an induced subgraph of the graph G, then $msr(H) \leq msr(G)$ as induced subgraphs of G correspond to principal submatrices of $A \in \mathcal{P}(G)$. The *compliment* of a graph G is the graph $\overline{G} = (V(G), \overline{E(G)})$, where $\overline{E(G)}$ consists of all two element sets from V that are not in $E(G)$.

1.2 Vector Representation

Given a set $\mathbf{V} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ of n vectors in $\mathbb{C}^m$ the matrix

$$\begin{pmatrix} \mathbf{v_1}^* \\ \mathbf{v_2}^* \\ \vdots \\ \mathbf{v_n}^* \end{pmatrix} \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{pmatrix}$$

is a PSD matrix called the *Gram matrix* of $\mathbf{V}$. Its associated graph G has n vertices $\{v_1, v_2, \ldots, v_n\}$ corresponding to the vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, and the edges of G correspond to the non-zero inner products among those vectors. Consequently, $\mathbf{V}$ is called a *vector representation* of G. For a set of vectors $\mathbf{V}$ we define $rank(\mathbf{V}) = dim(span(\mathbf{V}))$.

Conversely, any PSD matrix A may be factored as Y^*Y for some $Y \in M_n(\mathbb{C})$ with $rank(A) = rank(Y)$ and so A is a Gram matrix. This directly implies that if $\mathbf{V}$ is a vector representation of G, then $msr(G) \leq rank(\mathbf{V})$. This also implies that finding a PSD matrix representation of minimum rank and finding a vector representation of minimum rank are equivalent problems. This encourages restricting the more general Hermitian eigenvalue problem to the class of PSD matrices.

Vector representations are very useful for deriving the msr. For example consider the case when $msr(G) = 1$. If $msr(G) = 1$ and $\mathbf{V}$ is a vector representation of G of minimum rank then $\mathbf{V}$ consists of vectors that are scalar multiples of each other. As the inner product of two nonzero vectors, where one is a scalar multiple of the other, is nonzero, $\mathbf{V}$ corresponds to K_n. Therefore, $msr(G) = 1$ if and only if $G = K_n$. The next lemma is somewhat useful for graph classification and an easy result of vector representations.

Lemma 1.1. *Let G be a connected graph and $u, v \in V(G)$ such that u and v are duplicate vertices, that is $N[v] = N[u]$, then $msr(G) = msr(G - u)$.*

Proof. Let $\mathbf{V} = \{\mathbf{u}, \mathbf{v}, \mathbf{z_1}, \ldots, \mathbf{z_k}\}$ be a vector representation of G of minimum rank then $\mathbf{V} = \{\mathbf{v}, \mathbf{z_1}, \ldots, \mathbf{z_k}\}$ is a vector representation of $(G - u)$, therefore $msr(G -$

$u) \leq msr(G)$. Now let $\mathbf{V} = \{\mathbf{v}, \mathbf{z_1}, \ldots, \mathbf{z_k}\}$ be a vector representation of $(G-u)$ of minimum rank. Then $\{\mathbf{v}, \mathbf{v}, \mathbf{z_1}, \ldots, \mathbf{z_k}\}$ is a vector representation of G which implies that $msr(G) \leq msr(G-u)$. $\square$

2 Superposition of Graphs

In this section $\mathbb{F}^*$ will denote a field $\mathbb{F}$ without its additive identity.

Definition 2.1. *A graph G is a* superposition *of two graphs, G_1 and G_2, if G is obtained by identifying G_1 and G_2 at a set of vertices, keeping all edges that are present in either G_1 or G_2.*

Lemma 2.1. *Suppose that $\mathbf{v} \in \mathbb{C}^n$, $X = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\} \subset \mathbb{C}^n$, and $A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset \mathbb{C}^*$. Then for all but finitely many values $\beta \in \mathbb{R}^*$ we have $\langle \mathbf{v_i}, \beta\mathbf{v}\rangle \neq \alpha_i$ for $1 \leq i \leq k$.*

Proof. β can be any nonzero real value not in the set $\{\frac{\alpha_i}{\langle \mathbf{v_i}, \mathbf{v}\rangle} \mid \langle \mathbf{v_i}, \mathbf{v}\rangle \neq 0\}$. $\square$

Theorem 2.1. *Let G be modified from the superposition of G_1 and G_2 by the removal of the edges of a star forest common to both subgraphs. Then, $msr(G) \leq msr(G_1) + msr(G_2)$.*

This theorem is a generalization of a result found in M. Booth, et al. [4] where a single edge is removed.

Proof. We assume that two stars are being removed. As adding isolated vertices does not change the msr, let $V(G_1) = V(G_2) = V(G)$. Assume $msr(G_1) = k$ and $msr(G_2) = l$. Let $\{\mathbf{v_{1,1}}, \ldots, \mathbf{v_{1,n_1}}, \mathbf{v_{2,1}}, \ldots, \mathbf{v_{2,n_2}}, \mathbf{w_1}, \mathbf{w_2}, \mathbf{z_1}, \ldots, \mathbf{z_m}\} \subset \mathbb{C}^k$ be a vector representation of G_1 where $\{\mathbf{v_{i,j}}\}_{j=1}^{n_i}$ are the pendants of the i^{th} star, $\mathbf{w_i}$ is the center of the i^{th} star, and $\{\mathbf{z_i}\}_{i=1}^{m}$ corresponds to the remaining vertices. Likewise, define $\{\mathbf{v'_{1,1}}, \ldots, \mathbf{v'_{1,n_1}}, \mathbf{v'_{2,1}}, \ldots, \mathbf{v'_{2,n_2}}, \mathbf{w'_1}, \mathbf{w'_2}, \mathbf{z'_1}, \ldots, \mathbf{z'_m}\} \subset \mathbb{C}^l$ to be a vector representation of G_2. We claim there is a valid vector representation of G where the vertex $v_{i,j}$ is represented by $\alpha_{i,j}\mathbf{v_{i,j}} \oplus \mathbf{v'_{i,j}}$, w_i is represented by $\beta_i\mathbf{w_i} \oplus \mathbf{w'_i}$, and z_i is represented by $\gamma_i\mathbf{z_i} \oplus \mathbf{z'_i}$. For every $\beta_1 \in \mathbb{R}^*$ there is a unique $\alpha_{1,j}$ such that

$$0 = \langle \alpha_{1,j}\mathbf{v_{1,j}} \oplus \mathbf{v'_{1,j}}, \beta_1\mathbf{w_1} \oplus \mathbf{w'_1}\rangle = \alpha_{1,j}\beta_1\langle \mathbf{v_{1,j}}, \mathbf{w_1}\rangle + \langle \mathbf{v'_{1,j}}, \mathbf{w'_1}\rangle$$

namely $\alpha_{1,j} = -\frac{1}{\beta_1}\frac{\langle \mathbf{v'_{1,j}}, \mathbf{w'_1}\rangle}{\langle \mathbf{v_{1,j}}, \mathbf{w_1}\rangle}$ (note the edge $v_{1,j}w_1$ is in both $E(G_1)$ and $E(G_2)$ therefore both inner products are nonzero). Furthermore

$$\langle \alpha_{1,j}\mathbf{v_{1,j}} \oplus \mathbf{v'_{1,j}}, \alpha_{1,k}\mathbf{v_{1,k}} \oplus \mathbf{v'_{1,k}}\rangle = \alpha_{1,j}\alpha_{1,k}\langle \mathbf{v_{1,j}}, \mathbf{v_{1,k}}\rangle + \langle \mathbf{v'_{1,j}}, \mathbf{v'_{1,k}}\rangle \tag{1}$$

is a quadratic equation in terms of $\frac{1}{\beta_1}$. Therefore there is only a finite number of real numbers for which β_1 forces equation 1 to be zero if either of $\langle \mathbf{v_{1,j}}, \mathbf{v_{1,k}}\rangle$ and $\langle \mathbf{v'_{1,j}}, \mathbf{v'_{1,k}}\rangle$ are nonzero. Pick β_1 to be any real number not in this finite set.

Now we demonstrate the existence of a valid β_2. A similar argument shows that the connectivity between $v_{2,j}$ and $v_{2,k}$ is violated by only a finite number of real numbers. But β_2 is also restricted by connectivity with the first star. Say $\mathbf{x} = \mathbf{x_1} \oplus \mathbf{x_2}$ is a vector representation of a vertex x in the first star. If either of $\langle \mathbf{x_1}, \mathbf{w_2} \rangle$ or $\langle \mathbf{x_2}, \mathbf{w_2'} \rangle$ are nonzero then $\langle \mathbf{x}, \beta_2 \mathbf{w_2} \oplus \mathbf{w_2'} \rangle$ must be nonzero. As this inner product is a linear equation in terms of β_2 there are only a finite number of real numbers that force it to be zero. Likewise if either of $\langle \mathbf{x_1}, \mathbf{v_{2,j}} \rangle$ or $\langle \mathbf{x_2}, \mathbf{v_{2,j}'} \rangle$ are nonzero then $\langle \mathbf{x}, \alpha_{2,j} \mathbf{v_{2,j}} \oplus \mathbf{v_{2,j}'} \rangle$ must be nonzero. Again this inner product is a linear equation in terms of $\frac{1}{\beta_2}$ therefore only a finite number of real numbers force zero. Then β_2 can be all but a finite number of real numbers.

Finally $\gamma_1, \gamma_2, \ldots$ can be picked sequentially using the following procedure for γ_k. With the notation of Lemma 2.1 let $\mathbf{v} = \mathbf{z_k} \oplus \mathbf{0}$,

$$X = \{\alpha_{i,j} \mathbf{v_{i,j}} \oplus \mathbf{v_{i,j}'}\} \cup \{\mathbf{w_i} \oplus \beta \mathbf{w_i'}\} \cup \{\gamma_i \mathbf{z_i} \oplus \mathbf{z_i'}\}_{i=1}^{k-1}$$

and $A = \{-\langle \mathbf{x}, \mathbf{0} \oplus \mathbf{z_k'} \rangle \mid \mathbf{x} \in X\}$. By Lemma 2.1 there exists $\gamma_k \in \mathbb{R}^*$ such that if $\langle \mathbf{x}, \mathbf{0} \oplus \mathbf{z_k'} \rangle \neq 0$ then $\langle \mathbf{x}, \gamma_k \mathbf{z_k} \oplus \mathbf{0} \rangle \neq -\langle \mathbf{x}, \mathbf{0} \oplus \mathbf{z_k'} \rangle$. This implies that $\langle \gamma_k \mathbf{z_k} \oplus \mathbf{z_k'}, \mathbf{x} \rangle = 0$ for $\mathbf{x} \in X$ if and only if $\langle \mathbf{z_k} \oplus \mathbf{0}, \mathbf{x} \rangle = 0$ and $\langle \mathbf{0} \oplus \mathbf{z_k'}, \mathbf{x} \rangle = 0$. Thus, the edges not removed by the star forest pattern remain intact, and a valid vector representation for G exists in $k + l$ dimensions.

A similar argument can be made for the case of additional stars. □

We say that $G_1, \ldots, G_k$ *cover* G if each vertex of G is in at least one G_i and for every pair of adjacent vertices v, w of G, v and w are adjacent in at least one G_i. The cover $C_1, \ldots, C_k$ is *clique cover* of G if each of $C_1, \ldots, C_k$ is a clique of G. The *clique cover number* of G, denoted $cc(G)$, is the minimum value of k for which there is a clique cover $C_1, \ldots, C_k$ of G. Theorem 2.1 implies two important corollaries concerning graph covers.

Corollary 2.1. *If $G_1, \ldots, G_k$ cover G then*

$$msr(G) \leq \sum_{i=1}^{k} msr(G_i)$$

Corollary 2.2. *For any graph G, $msr(G) \leq cc(G)$.*

In Booth et al. [3] it is shown that the $msr(G) = cc(G)$ for all chordal graphs, which are graphs without an induced cycle of more than four vertices.

3 Orthogonal Vertex Removal

Orthogonal removal of vertices is an idea first introduced by Booth et al. [3]. The notion has proven quite useful not only in finding the msr of specific graphs, but also in proving results about the msr of graphs in general.

If G is an undirected graph on n vertices that has no loops but may have multiple edges, then $\mathcal{P}(G)$ denotes the set of all PSD matrices that relate to a common graph, G, where

- $a_{ij} \neq 0$ if i and j are connected by exactly one edge,
- $a_{ij} = 0$ if i and j are not adjacent and $i \neq j$, and
- no restriction is made on a_{ij} if i and j are connected by more than one edge.

If G is a multigraph, or a graph where more than one edge is possible between to vertices, we define H to be as a *minimum graph representation* of G, simple graph $\mathcal{P}(H) \subset \mathcal{P}(G)$, and $msr(G) = msr(H)$. By definition of the msr of a multigraph it is clear that every multigraph has at least one minimum graph representation. Furthermore, if H is a minimum graph representation of G, then H will be the graph G where every multiedge is either removed or replaced by a single edge.

If $v \in V(G)$, the multigraph $G \ominus v$ is defined to be the induced subgraph on vertices $V(G) - v$ such that any $u, w \in N(v)$ have $e - 1$ edges added between them, where e is the sum of the number of edges from v to u and from v to w.

Remark 3.1. *Let* $\mathbf{V}$ *be a vector representation of a graph* G*. If* G *contains a vertex* v *with corresponding vector* $\mathbf{v}$ *then define* $\mathbf{V} \ominus \mathbf{v}$ *to be the set of vectors*

$$\mathbf{w} - \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

where $\mathbf{w} \in \mathbf{V}$ *and* $\mathbf{w} \neq \mathbf{v}$*. Then as* $\mathbf{V} \ominus \mathbf{v}$*, is a vector representation of* $G \ominus v$ *and the dimension of* $\mathbf{V}$ *is one more than the dimension of* $\mathbf{V} \ominus \mathbf{v}$ *it is clear that* $msr(G \ominus v) + 1 \leq msr(G)$*.*

With Remark 3.1 it becomes desirable to find cases where $msr(G) = msr(G \ominus v) + 1$. The next corollary was originally proven in P. Hackney et al. [5] and generalizes all known results about when $msr(G) = msr(G \ominus v) + 1$.

Corollary 3.1. *Let* G *be a connected graph with vertex* v*, and let* H *be the graph induced by the vertices of* $N(v)$*. If* H *is a simple graph and* $\overline{H}$ *is a star forest, then* $msr(G) = msr(G \ominus v) + 1$*.*

Proof. By Remark 3.1 it is enough to demonstrate that $msr(G) \leq msr(G \ominus v) + 1$. Theorem 2.1 will be employed to show this. Let $m = |N(v)| + 1$. Set $G_1 = K_m$ and let G_2 be a minimum graph representation of $G \ominus v$. Let the vertices of G_1 correspond to the vertices of $N[v]$. Then $msr(G) \leq msr(G_1) + msr(G_2) = 1 + msr(G \ominus v)$ as G can be modified from the superposition of G_1 and G_2 by the removal of a star forest - namely, $\overline{H}$. $\square$

Two important special cases of Corollary 3.1 are given as lemmas.

Lemma 3.1. *Let G be a connected graph and v a vertex of degree one. Then $msr(G) = msr(G \ominus v) + 1$.*

Note that $G \ominus v = G - v$ if $deg(v) = 1$.

Lemma 3.2. *Let G be a connected graph and v a vertex of degree two. Then $msr(G) = msr(G \ominus v) + 1$.*

A proof of Lemma 3.1 is shown in [1] and a proof of Lemma 3.2 is shown in [3].

4 Vertex Sum of Graphs

In this section we present a new result on msr of vertex sum of graphs and give a simpler proof of a prior result on vertex sum.

The following lemma is a straightforward result that will be useful in proving our theorems.

Lemma 4.1. *Let $span(X) = span(\{\mathbf{v_i}\} \cup \{\mathbf{w_i}\})$ where $\langle \mathbf{v_i}, \mathbf{w_j} \rangle = 0$ for all i, j. Then $dim(span(X)) = dim(span(\{\mathbf{v_i}\})) + dim(span(\{\mathbf{w_i}\}))$.*

Proof. It is clear that $dim(span(X)) \leq dim(span(\{\mathbf{v_i}\})) + dim(span(\{\mathbf{w_i}\}))$. Let $\{\mathbf{v_{s_i}}\}$ be a basis for $span(\{\mathbf{v_i}\})$ and $\{\mathbf{w_{t_i}}\}$ be a basis for $span(\{\mathbf{w_i}\})$. We claim that $\{\mathbf{v_{s_i}}\} \cup \{\mathbf{w_{t_i}}\}$ is a linearly independent set. Assume that there exist nonzero scalars α_i and β_i such that $\sum \alpha_i \mathbf{v_{s_i}} + \sum \beta_i \mathbf{w_{t_i}} = \mathbf{0}$. Then $\sum \alpha_i \mathbf{v_{s_i}} = -\sum \beta_i \mathbf{w_{t_i}} \neq \mathbf{0}$ (as $\{\mathbf{v_i}\}$ is a basis and there is some nonzero α scalar) which implies that $\langle \sum \alpha_i \mathbf{v_{s_i}}, -\sum \beta_i \mathbf{w_{t_i}} \rangle > 0$ which is impossible as $\langle \mathbf{v_i}, \mathbf{w_j} \rangle = 0$ for all i, j. Thus $\{\mathbf{v_{s_i}}\} \cup \{\mathbf{w_{t_i}}\}$ is a linearly independent set. This demonstrates that $dim(span(X)) \geq dim(span(\{\mathbf{v_i}\})) + dim(span(\{\mathbf{w_i}\}))$. $\square$

Definition 4.1. *A vertex v of a connected graph G is said to be a* cut vertex *of G if the subgraph of G induced by the removal of v is disconnected.*

Definition 4.2. *Suppose G is decomposable into two connected graphs, G_1 and G_2, sharing only one vertex v such that if $u \in V(G_1)$ and $w \in V(G_2)$, then $uw \in E(G)$ only if $u = v$ or $w = v$. Then G_1 and G_2 are joined at a* cut vertex, *and we write $G = G_1 \centerdot G_2$.*

Theorem 4.1. *Let $G = G_1 \centerdot G_2$ then $msr(G) = msr(G_1) + msr(G_2)$.*

This theorem was originally proved in [4] using row and column operations of a positive semidefinite matrix of minimum rank. Here we give a simpler proof by vector representations.

Proof. As G is the superposition of G_1 and G_2 using Theorem 2.1 we get $msr(G) \leq msr(G_1) + msr(G_2)$. Let $msr(G) = m$ and $\{\mathbf{w_i}\} \cup \{\mathbf{v}\} \cup \{\mathbf{z_i}\} \subset \mathbb{C}^m$ be a vector representation of G where $\{\mathbf{w_i}\} \cup \{\mathbf{v}\}$ is a vector representation of G_1 and $\{\mathbf{z_i}\} \cup \{\mathbf{v}\}$ is a vector representation of G_2. It is clear that $\langle \mathbf{w_i}, \mathbf{z_j} \rangle = 0$ for all i, j. Since the vertex v is adjacent to vertices in G_1 and G_2 the vector $\mathbf{v}$ can be rewritten as

$$\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{v}'$$

where $\mathbf{a} \in span(\{\mathbf{w_i}\})$, $\mathbf{b} \in span(\{\mathbf{z_i}\})$, and $\mathbf{v}'$ is orthogonal to the vectors in the set $\{\mathbf{w_i}\} \cup \{\mathbf{z_i}\}$. Then $\{\mathbf{w_i}\} \cup \{\mathbf{a} + \mathbf{b}\} \cup \{\mathbf{z_i}\} \subset \mathbb{C}^m$ is a vector representation of G of minimum rank, $X = \{\mathbf{w_i}\} \cup \{\mathbf{a}\}$ is a vector representation of G_1, and $Y = \{\mathbf{z_i}\} \cup \{\mathbf{b}\}$ is a vector representation of G_2. Furthermore $span(\{\mathbf{w_i}\} \cup \{\mathbf{a} + \mathbf{b}\} \cup \{\mathbf{z_i}\}) = span(\{\mathbf{w_i}\} \cup \{\mathbf{z_i}\}) = span(X \cup Y)$. Then by Lemma 4.1, $msr(G) = dim(span(X)) + dim(span(Y)) \geq msr(G_1) + msr(G_2)$. □

Definition 4.3. *Let G be a graph that does not contain a cut vertex. Vertices u and v of G are said to be* double cut vertices *if $uv \notin E(G)$ and the subgraph of G induced by the removal of u and v is disconnected.*

Definition 4.4. *Suppose G is decomposable into two connected graphs G_1 and G_2 sharing only two vertices u and v such that if $x \in V(G_1)$ and $y \in V(G_2)$, then $xy \in E(G)$ only if x or y is equal to u or v. Then G_1 and G_2 are joined at double cut vertices and we write $G = G_1 \star G_2$. Also define G_i' to be a graph with vertex set $V(G_i)$ and edge set $E(G_i) \cup \{uv\}$.*

Theorem 4.2. *If $G = G_1 \star G_2$ then*

$$msr(G) = \min\{msr(G_1) + msr(G_2), msr(G_1') + msr(G_2')\}.$$

This proof is very similar to the argument given in Theorem 4.1.

Proof. Since G is the superposition of G_1 and G_2 and G is the superposition of G_1' and G_2' minus an edge, Theorem 2.1 implies that $msr(G) \leq \min\{msr(G_1) + msr(G_2), msr(G_1') + msr(G_2')\}$. Let $msr(G) = m$ and $\{\mathbf{w_i}\} \cup \{\mathbf{u}, \mathbf{v}\} \cup \{\mathbf{z_i}\} \subset \mathbb{C}^m$ be a vector representation of G where $\{\mathbf{w_i}\} \cup \{\mathbf{u}, \mathbf{v}\}$ is a vector representation of G_1 and $\{\mathbf{z_i}\} \cup \{\mathbf{u}, \mathbf{v}\}$ is a vector representation of G_2. It is clear that $\langle \mathbf{w_i}, \mathbf{z_j} \rangle = 0$ for all i, j. The vectors $\mathbf{u}$ and $\mathbf{v}$ can be rewritten as

$$\mathbf{u} = \mathbf{a} + \mathbf{b} + \mathbf{u}'$$
$$\mathbf{v} = \mathbf{c} + \mathbf{d} + \mathbf{v}'$$

where $\mathbf{a}, \mathbf{c} \in span(\{\mathbf{w_i}\})$; $\mathbf{b}, \mathbf{d} \in span(\{\mathbf{z_i}\})$; and the vectors $\mathbf{u}'$ and $\mathbf{v}'$ are orthogonal to the vectors in the set $\{\mathbf{w_i}\} \cup \{\mathbf{z_i}\}$. As $0 = \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b} + \mathbf{u}', \mathbf{d} + \mathbf{v}' \rangle$ we have $\langle \mathbf{a}, \mathbf{c} \rangle = -\langle \mathbf{b} + \mathbf{u}', \mathbf{d} + \mathbf{v}' \rangle$.

Case 1: Assume $\langle \mathbf{a}, \mathbf{c} \rangle = 0$. Set $X = \{\mathbf{w_i}\} \cup \{\mathbf{a}, \mathbf{c}\}$ and $Y = \{\mathbf{z_i}\} \cup \{\mathbf{b} + \mathbf{u}', \mathbf{d} + \mathbf{v}'\}$. Then X and Y are vector representations of G_1 and G_2 respectively. Furthermore $\mathit{span}(\{\mathbf{w_i}\} \cup \{\mathbf{a}+\mathbf{b}+\mathbf{u}', \mathbf{c}+\mathbf{d}+\mathbf{v}'\} \cup \{\mathbf{z_i}\}) = \mathit{span}(\{\mathbf{w_i}\} \cup \{\mathbf{z_i}\} \cup \{\mathbf{u}', \mathbf{v}'\}) = \mathit{span}(X \cup Y)$. Then by Lemma 4.1, $\mathit{msr}(G) = \mathit{dim}(\mathit{span}(X)) + \mathit{dim}(\mathit{span}(Y)) \geq \mathit{msr}(G_1) + \mathit{msr}(G_2)$.

Case 2: Assume $\langle \mathbf{a}, \mathbf{c} \rangle \neq 0$. Set $X = \{\mathbf{w_i}\} \cup \{\mathbf{a}, \mathbf{c}\}$ and $Y = \{\mathbf{z_i}\} \cup \{\mathbf{b} + \mathbf{u}', \mathbf{d} + \mathbf{v}'\}$. Then X and Y are vector representations of G_1' and G_2' respectively. Then by Lemma 4.1, $\mathit{msr}(G) = \mathit{dim}(\mathit{span}(X)) + \mathit{dim}(\mathit{span}(Y)) \geq \mathit{msr}(G_1') + \mathit{msr}(G_2')$. ☐

5 $\mathit{msr}(G) = |G| - 2$

The classification of all graphs having msr two less than their order was first completed by Holst [1]. In this section we apply the power of orthogonal removal to demonstrate a different method of classification. In many ways the classification presented is similar to the Johnson et al. classification of all graphs G having $mr(G) = |G| - 2$ [2], where $mr(G)$ vectors to real symmetric matrices instand of PSD matrices.

A connected graph G is *k-connected* if the removal of any $k-1$ vertices results in a connected graph.

Theorem 5.1. *Let G be graph. Then G is 1-connected graph and $\mathit{msr}(G) = |G| - 2$ if and only if*

$$G = G_1 \centerdot (G_2 \centerdot (G_3 \centerdot (\dots (G_k \centerdot H) \dots)))$$

where $G_1, G_2, \dots G_k$ are trees and H is a 2-connected graph such that $\mathit{msr}(H) = |H| - 2$.

Proof. ($\Rightarrow$) As G is 1-connected $G = G_1 \centerdot H$ for $G_1, H \subset G$. Assume that neither of G_1 and H are trees then $\mathit{msr}(G_1) \leq |G_1| - 2$ and $\mathit{msr}(H) \leq |H| - 2$. This directly implies via Theorem 4.1 that $\mathit{msr}(G) = \mathit{msr}(G_1) + \mathit{msr}(H) \leq |G| - 3$ as $|G_1| + |H| = |G| + 1$. Assume that G_1 is a tree then as $|G| - 2 = \mathit{msr}(G) = \mathit{msr}(G_1) + \mathit{msr}(H)$ we have $\mathit{msr}(H) = |H| - 2$. This directly implies that $G = G_1 \centerdot (G_2 \centerdot (G_3 \centerdot (\dots (G_k \centerdot H) \dots)))$ where $G_1, G_2, \dots G_k$ are trees and H is a 2-connected graph such that $\mathit{msr}(H) = |H| - 2$.

($\Leftarrow$) This direction is a straight forward result of Theorem 4.1. For observe,

$$\begin{aligned} \mathit{msr}(G) &= \mathit{msr}(G_1) + \mathit{msr}(G_2) + \cdots + \mathit{msr}(G_k) + \mathit{msr}(H) \\ &= |G_1| - 1 + |G_2| - 1 + \cdots + |G_k| - 1 + |H| - 2 \\ &= |G| - 2 \end{aligned}$$

as $|G_1| + |G_2| + \cdots + |G_k| + |H| = |G| + k$. ☐

Now the problem of classification is restricted to 2-connected graphs. Before addressing this restricted problem, we will go over some definitions. A *k-tree* is a graph sequentially built from $k+1$-cliques (K_{k+1}) via articulation along k-cliques. A *partial k-tree* is a k-tree from which some edges (without incident vertices) have been deleted. Note that a partial 2-tree always has a vertex of degree one or two.

Theorem 5.2. *([1]) If a graph G is not a partial 2-tree, then $msr(G) \leq |G| - 3$.*

Corollary 5.1. *Let G be a 2-connected graph. If $msr(G) = |G| - 2$ then G is a partial 2-tree and G contains a vertex of degree 2.*

Proof. By Theorem 5.2 G must be a partial 2-tree. This implies that G contains either a vertex of degree one or two. If G contains a vertex of degree one then G is not 2-connected. □

The graph in Figure 9 is a chordal partial 2-tree. Its msr less than or equal to 3 which is the clique cover number(Corollary 2.2). Thus, there exist partial 2-trees with msr less than $|G| - 2$.

Definition 5.1. *A graph G is an $MPAC$ (multiple path-articulated cycle) graph if it is sequentially built from cycles via edge subdivision or articulation of each successive cycle along a path on the previous graph. Each articulation must add at least one new vertex to the graph.*

$MPAC$ graphs are a generalization of Johnson et al. [2] definition of $SEAC$ (singly edge-articulated cycle) graphs. A graph G is a $SEAC$ graph if it is sequenticially built from cycles via articulation of each successive cycle along an edge on the previous graph. No such edge may be used more than once for articulation.

Theorem 5.3. *Let G be a 2-connected graph. If $msr(G) = |G| - 2$ then G is a $MPAC$ graph.*

Proof. If $|G| = 3$ then $G = K_3$, $msr(G) = 1$ and K_3 is a $MPAC$ graph. Assume $|G| = n > 3$. By Corollary 5.1, G has a vertex w such that $deg_G(w) = 2$ and $N(w) = \{u, v\}$. Let $G_{n-1} = (V(G) - w, E(G) \cup \{uv\})$. As $msr(G \ominus w) + 1 = msr(G)$, we have $msr(G_{n-1}) + 1 \geq msr(G) = n - 2$. Also $msr(G_{n-1}) \leq |G_{n-1}| - 1 = n - 2$. Therefore $msr(G_{n-1}) = n - 2$ or $n - 3$. If $msr(G_{n-1}) = n - 2$ then G_{n-1} is a tree. Since every tree must have at least two pendant vertices it follows that $deg_{G_{n-1}}(v) = 1$ and $deg_{G_{n-1}}(u) = 1$ as u and v are the only vertices affected by the removal of w. Since $|G| > 3$ and $uv \in E(G_{n-1})$ it follows that G was disconnected by the removal of a single vertex which is a contradiction as G is a 2-connected graph. Thus $msr(G_{n-1}) = n - 3 = |G_{n-1}| - 2$ and it is easy to see that G_{n-1} is 2-connected.

Repeating the same procedure successively we form $G_{n-2}, G_{n-3}, \ldots, G_3$ where $msr(G_i) = |G_i| - 2 = i - 2$. As G_3 is 2 connected we must have $G_3 = K_3$. Now we add back the removed vertices to form $G_4, G_5, \ldots, G_{n-1}, G$. In each step we follow the procedure found in the definition of a $MPAC$ graph. □

The graph in Figure 9 is a chordal $MPAC$ graph. Its msr less than or equal to 3, which is the clique cover number [3]. Thus, there exist $MPAC$ graphs with msr less than $|G| - 2$.

We say that two cycles are *neighbors* if they share a path of articulation. The endpoints of the path of articulation will be referred to as the *points of articulation*.

Definition 5.2. *A $MPAC$ graph is said to be $RMPAC$ (restricted $MPAC$) if it satisfies the following two conditions:*

1. *The points of articulation of each successive cycle are vertices on one of the previously added cycles.*
2. *Suppose $C_1, C_2, \dots, C_k$ are the neighbors of C. Then, there is a simple path P on C consisting of the points of articulation $\{u_{C_i}, v_{C_i}\}$, $1 \leq i \leq k$, in the following order: $u_{C_{n_1}}, u_{C_{n_2}}, \dots, u_{C_{n_k}}, v_{C_{n_k}}, \dots, v_{C_{n_2}}, v_{C_{n_1}}$ where $\{n_1, \dots, n_k\}$ is some ordering of $\{1, \dots, k\}$. It is not necessary that the points of articulation be distinct. The path P is called a* neighbor path *of C and $C, C_{n_1}, C_{n_2}, \dots, C_{n_k}$ form a* "deck of cycles"*. Moreover C_{n_k} is called a* inner cycle *of C.*

RMPAC graphs are a generalization of Johnson et al. [2] definition of linear $SEAC$ ($LSEAC$) graphs. A graph G is a $LSEAC$ graph if each of its constituent cycles has at most two neighbors. Johnson et al. prove that G is a 2-connected graph such that $mr(G) = 2$ if and only if G is a $LSEAC$ graph.

Remark 5.1. *If C has a neighbor path P with the ordering $u_{C_{n_1}}, u_{C_{n_2}}, \dots, u_{C_{n_k}}, v_{C_{n_k}}, \dots, v_{C_{n_2}}, v_{C_{n_1}}$ then there is a neighbor path P' with the ordering $u_{C_{n_k}}, u_{C_{n_{k-1}}}, \dots, u_{C_{n_1}}, v_{C_{n_1}}, \dots, v_{C_{n_{k-1}}}, v_{C_{n_k}}$.*

Lemma 5.1. *Let G be an $RMPAC$ graph. G must contain at least one cycle with only two vertices of degree greater than two.*

Proof. If C is a cycle with a neighbor path with ordering $u_{C_1}, u_{C_2}, \dots, u_{C_k}, v_{C_k}, \dots, v_{C_2}, v_{C_1}$, then C_k is a cycle with the desired properties if it has no neighbors other than $C, C_1, \dots, C_{k-1}$. An $RMPAC$ graph can be thought of as a list of cycles: $G \approx C_1 C_2 \dots C_n$ where C_i is the i^{th} cycle joined to the graph. If $n = 2$ then both cycles have only two vertices of degree greater than two. If $n > 2$, then select a cycle C with at least two neighbors. Let $u_{C_1}, u_{C_2}, \dots, u_{C_k}, v_{C_k}, \dots, v_{C_2}, v_{C_1}$ be the ordering of a neighbor path of C. If C_k has a neighbor other than $C, C_1, \dots, C_{k-1}$, consider the two inner cycles of C_{k-1} guaranteed by Remark 5.1. At least one of these inner cycles is not equal to $C, C_1, \dots, C_{k-1}$. Repeat this procedure, each time finding at least one more distinct inner cycle. As G has a only a finite number of cycles, this procedure must terminate with a cycle with the desired properties. □

Theorem 5.4. *Let G be an $RMPAC$ graph. G must be 2-connected and $msr(G) = |G| - 2$.*

Proof. G is clearly 2-connected. Let $G \approx C_1 C_2 \dots C_x$. If $x = 1$ or 2 then $msr(G) = |G| - 2$. Induct on x. If $x > 2$ use Lemma 5.1 to consider a cycle C_i that has only two vertices of degree greater than two. Remove all vertices of degree two on the cycle C_i to form G' where G' has $x - 1$ cycles, and G' has a single multi edge between

$u_{C_i}v_{C_i}$ where u_{C_i} and v_{C_i} are the points articulation of the cycle C_i. By Lemma 3.2 $msr(G) = msr(G') + |G| - |G'|$. We consider the graphs G_1 and G_2 where G_1 is equal to G' but with a single edge between vertices u_{C_i} and v_{C_i} and G_2 is equal to G' but with no edge between vertices u_{C_i} and v_{C_i}. By definition $msr(G') = \min\{msr(G_1), msr(G_2)\}$. G_1 is a $RMPAC$ graph with $x - 1$ cycles so $msr(G_1) = |G_1| - 2$ by our induction hypothesis. In G_2 consider a neighbor of C_i in G, C_j. C_j has a neighbor path with ordering $u_{C_{n_1}}, u_{C_{n_2}}, \ldots, u_{C_{n_k}}, v_{C_{n_k}}, \ldots, v_{C_{n_2}}, v_{C_{n_1}}$ where $n_k = n_i$. Clearly $k \geq 2$ as $x > 2$ and G is connected. Now we remove the vertices between $u_{C_{n_k}}$ and $u_{C_{n_{k-1}}}$ and the vertices between $v_{C_{n_k}}$ and $v_{C_{n_{k-1}}}$ to form G'_2. If these vertices are removed in a sequential manner each vertex removed will be a pendant. Then by Lemma 3.1 $msr(G'_2) + |G_2| - |G'_2| = msr(G_2)$. G'_2 is a $RMPAC$ graph where the remaining vertices on cycle C_j and $C_{n_{k-1}}$ form a cycle. Then $msr(G'_2) = |G'_2| - 2$ by the induction hypothesis. The relationship between $msr(G), msr(G'), msr(G_1), msr(G_2)$, and $msr(G'_2)$ implies that $msr(G) = |G| - 2$. □

Lemma 5.2. *Let H be a $RMPAC$ graph. Form G by adding a cycle via articulation to vertices u and v in H in a manner that violates property 1 of the $RMPAC$ definition. Then $msr(G) < |G| - 2$.*

Proof. Clearly G is not a tree so $msr(G) \leq |G| - 2$. Assume that $msr(G) = |G| - 2$. Furthermore let $u \in C_i$ and $v \in C_j$. Assume that C_i and C_j are neighbors. The following argument is easily generalized to the case where C_i and C_j are connected via a series of adjacent cycles. Let x and y be the points of articulation between C_i and C_j. Then there is a set of disjoint paths connecting each of $\{u, v, x, y\}$. Use the procedure of Theorem 5.3 to form $G_{n-1}, G_{n-2}, \ldots, G_3$. The disjoint paths are perserved with each degree 2 vertex removal, though they will be shortened however. This directly implies that if $a \in \{u, v, x, y\}$ and $deg_{G_i}(a) = 2$ then $b \notin V(G_i)$ for some $b \in \{u, v, x, y\}/\{a\}$. Therefore it is impossible to remove any of these four vertices, which is a contradiction as Theorem 5.3 demonstrates that reduction to three vertices is possible. □

Lemma 5.3. *Let H be an $RMPAC$ graph. Form G by adding a cycle via articulation to vertices u and v in H in a manner that violates property 2 of the $RMPAC$ definition. Then $msr(G) \neq |G| - 2$.*

Proof. Because of Lemma 5.2 we assume that the vertices u and v do not violate property 1 of the $RMPAC$ definition. Let C be a previously added cycle that contains u and v. Let P be a simple path that contains all the points of articulation of the neighbors of C. As P cannot be of the form given in property 2 of the $RMPAC$ definition there are two possiblities for the ordering of the points of articulation. The first case is when C has a path with ordering $u_{C_1}, \ldots, u_{C_2}, \ldots, v_{C_1}, \ldots, v_{C_2}$ where each articulation point is distinct. In this case, an argument identical to Lemma 5.2 can be employed using the vertices $\{u_{C_1}, u_{C_2}, v_{C_1}, v_{C_2}\}$.

The second case is an ordering of the form $u_{C_1}, \ldots, v_{C_1}, \ldots, u_{C_2}, \ldots, v_{C_2}, \ldots, u_{C_3}, \ldots, v_{C_3}$. Consider P in H. Assume that there is a cycle C_4 with articulation points on P such that the ordering $u_{C_4}, \ldots, u_{C_1}, \ldots, v_{C_1}, \ldots, u_{C_2}, \ldots, v_{C_2}, \ldots, v$ is achieved. This ordering can yield no valid neighbor path on C, which is a contradiction, as H is an $RMPAC$ graph. Then, without loss of generality, let C_1 and C_2 be cycles such that there are no articulation points between the beginning of P and u_{C_1}, between v_{C_1} and u_{C_2} on P, and between v_{C_2} and the end of P. We assume that $msr(G) = |G| - 2$ and follow the procedure of Theorem 5.3 to form $G_{n-1}, G_{n-2}, \ldots, G_k$ where $msr(G_i) = |G_i| - 2 = i - 2$ and $msr(G) = msr(G_i) + |G| - i$. A vertex w is not removed, however, if $N(w) = \{x, y\}$ where x and y are vertices on the cycle C. Assume the procedure terminates at G_k. Then G_k is the cycle C with some number of vertices of degree two attached. Because of the definition of C_1 and C_2 we can view the vertices on P between u_{C_1} and v_{C_1}; between u_{C_2} and v_{C_2}, and between u_{C_3} and v_{C_3} can be viewed as three "deck of cycles" on C. Using Lemma 3.2 each deck can be reduced down to a single multiedge between u_{C_i} and v_{C_i} call this graph G'_k. Then G'_k is a cycle with three multiedges and $msr(G_k) = msr(G'_k) + |G_k| - |G'_k|$. As the multiedges can divide G'_k into three connected components $msr(G'_k) \leq |G'_k| - 3$. With this relationship between $msr(G'_k)$, $msr(G_k)$, and $msr(G)$ it is clear that $msr(G) \leq |G| - 3$. $\square$

Theorem 5.5. *G is a 2 connected graph and $msr(G) = n - 2$ if and only if G is an $RMPAC$ graph.*

Proof. Use Lemma 5.2 and 5.3 to make the reconstruction phase of Theorem 5.3 stronger. $\square$

6 Graph Classes

In this section the msr of several families of graphs is determined. The families considered closely models the work done in [7].

result #	G	order	$mr(G)$	$msr(G)$
6.2	$K_s \Box P_t$	st	$s(t-1)$	$s(t-1)$
6.3	T_n(supertriangle)	$\frac{1}{2}n(n+1)$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}n(n-1)$
6.4	$G \circ H$	$\lvert G\rvert \lvert H\rvert + \lvert G\rvert$		$msr(G) + \lvert G\rvert\, msr(H)$
6.5	$P_s \boxtimes P_t$	st	$(s-1)(t-1)$	$(s-1)(t-1)$
6.6	Petersen	10	5	6
6.7	Q_n (hypercube)	2^n	2^{n-1}	2^{n-1}
6.8	$L(T)$m T a tree and $l = \#$ of pendents vertices of T	$\lvert T\rvert - 1$	$\lvert T\rvert - l$	$\lvert T\rvert - l$
6.9	$L(K_n)$	$\frac{1}{2}n(n-1)$	$n-2$	$n-2$
6.10	$K_s \Box K_t$	st	$s+t-2$	$s+t-2$
6.5, 6.6	$L(G)$ (if $\lvert G\rvert = n$) if G has a Hamiltonian path or contains $K_{k,n-k}$ as a subgraph $(1 < k < n)$		$n-2$	$n-2$

We now consider a definition from [5], which proves to be a powerful tool in determining the $msr(G)$.

Definition 6.1. *Let G be a connected graph and let $S = \{v_1, \ldots, v_m\}$ be an ordered set of vertices of G. Denote by G_k the subgraph induced by $v_1, \ldots, v_k$ for each $k, 1 \leq k \leq m$. Let H_k be the connected component of G_k containing v_k. If for each k, $1 \leq k \leq m$ there exists $w_k \in V(G), w_k \neq v_l$ for $l \leq k, (w_k, v_k) \in E(G), (w_k, v_s) \notin E(G)$ for all $v_s \in V(H_k)$ with $s \neq k$, then S is called an* OS-vertex set.

Definition 6.2. *The OS number of G, denoted $OS(G)$, is the size of the maximum OS-vertex set of the graph, G.*

Theorem 6.1. *([5]) For any graph G,*

$$OS(G) \leq msr(G)$$

Theorem 6.2. *For $t > 1$ $msr(K_s \Box P_t) = s(t-1)$*

Proof. $V(K_s \Box P_t) = \cup_{i=1}^{t} \{v_{i,j}\}_{j=1}^{s}$ where $\{v_{i,j}\}_{j=1}^{s}$ is the vertices in a copy of K_s. It is clear that $\cup_{i=1}^{t-1} \{v_{i,j}\}_{j=1}^{s}$ is a valid OS-set of $K_s \Box P_t$ thus by Theorem 6.1 $s(t-1) \leq msr(K_s \Box P_t)$. To determine an upperbound we induct on t. When $t = 2$ a valid vector representation of $K_s \Box P_2$ in $\mathbb{C}^s$ can be found. Let $\mathbf{v_{i,j}}$ represent the vertex $v_{i,j}$. Then set

$$\mathbf{v_{1,j}} = \mathbf{e_j} + \sum_{k=1}^{s} \mathbf{e_k}$$

$$\mathbf{v_{2,j}} = (s+1)\mathbf{e_j} - \sum_{k=1}^{s} \mathbf{e_k}$$

It is clear that $\cup_{i=1}^{2} \{\mathbf{v_{i,j}}\}_{j=1}^{s}$ is a vector repersentation of $K_s \Box P_2$ in $\mathbb{C}^s$. This implies that $msr(K_s \Box P_2) \leq s$. Now consider t where the upperbound is true for $k < t$. It is clear that the graphs $K_s \Box P_2$ and $K_s \Box P_{t-1}$ can cover the edges of $K_s \Box P_t$. Then Theorem 2.1 implies that $msr(K_s \Box P_t) \leq msr(K_s \Box P_{t-1}) + msr(K_s \Box P_2) \leq s(t-2) + s = s(t-1)$. $\Box$

Definition 6.3. *The n^{th}* supertriangle*, T_n, is an equilateral triangular grid with n vertices on each side.*

Theorem 6.3. $msr(T_n) = \frac{n(n-1)}{2}$

Proof. The edges of T_n can be covered by $\frac{n(n-1)}{2}$ triangles. Then Theorem 2.2 implies that $msrT_n \leq \frac{n(n-1)}{2}$. The set of vertices of an embedded T_{n-1} form a valid OS-Set of size $\frac{n(n-1)}{2}$ in T_n. Therefore, $msr(T_n) \geq \frac{n(n-1)}{2}$. $\Box$

The *corona* of a graph G with a graph H, denoted $G \circ H$, is the graph of order $|G|\,|H| + |G|$ obtained by taking one copy of graph G and $|G|$ copies of H, and joining all vertices in the i^{th} copy of H to the i^{th} vertex of G. For example see Figure [REF].

Theorem 6.4.

$$msr(G \circ H) = msr(G) + |G|\, msr(H)$$

Proof. $G \circ H$ can be written as a series of vertex sums. For example when $|G| = 3$,

$$G \circ H = (H \vee v) \centerdot ((H \vee v) \centerdot ((H \vee v) \centerdot G)).$$

Then Theorem 4.1 implies that $msr(G \circ H) = msr(G) + |G|\, msr(H \vee v)$. Furthermore, since $msr(H \vee v) = msr(H)$ if H is connected we have

$$msr(G \circ H) = msr(G) + |G|\, msr(H).$$

$\square$

Knowing that, for T a tree, $msr(T) = n - 1$, $msr(C_n) = n - 2$, and $msr(K_n) = 1$ allows for Theorem 6.4 to be given in three special cases:

Corollary 6.1. $msr(G \circ K_n) = msr(G) + |G|$

Corollary 6.2. $msr(G \circ C_n) = msr(G) + |G|\,(n - 2)$

Corollary 6.3. $msr(G \circ T) = msr(G) + |G|\,(n - 1)$

A *strong product* of graphs G and H, denoted $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if one of the following holds true:

1. $uu' \in E(G)$
2. $u = u'$ and $vv' \in E(H)$
3. $v = v'$ and $uu' \in E(G)$

Theorem 6.5.

$$msr(P_s \boxtimes P_t) = (s - 1)(t - 1)$$

Proof. Because the edges of $P_s \boxtimes P_t$ can be covered by $(s - 1)(t - 1)$ copies of K_4, Theorem 2.2 implies that $msr(P_s \boxtimes P_t) \leq (s - 1)(t - 1)$. To find a lower bound, an OS-set of size $(s-1)(t-1)$ will be demonstrated. Let $v_{i,j}$ where $1 \leq i \leq s$,$1 \leq j \leq t$ be the vertices of $P_s \boxtimes P_t$. The set $v_{i,j}$ where $1 \leq i \leq s - 1$,$1 \leq j \leq t - 1$ is a valid OS-set where the vertices are picked in the order $v_{1,1}, v_{2,1}, \ldots, v_{(s-1),1}, v_{2,2}, \ldots, v_{(s-1),2}, \ldots$ and the w corresponding to $v_{i,j}$ is $v_{i+1,j+1}$. Then Theorem 6.1 implies that $msr(P_s \boxtimes P_t) \geq (s - 1)(t - 1)$. For an example of this OS-set pattern, see Figure [REF]. $\square$

Theorem 6.6. *Let G be the Peterson graph defined by Figure [REF], then $msr(G) = 6$.*

Proof. There is a valid OS-set of six vertices therefore $6 \leq msr(G)$. Consider the matrix

$$M = A(G) + 2I = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$

where $A(G)$ is the adjacency matrix of G. Direct inspection shows that $rank(M) = 6$ and M is PSD. Then $msr(G) \leq 6$. □

Theorem 6.7. $msr(Q_n) = 2^{n-1}$

Proof. Since the $msr(Q_n)$ is bounded below by the size of the maximum independent set, we know that $2^{n-1} \leq msr(Q_n)$. So, all that is needed is a complex 2^{n-1}-dimensional vector representation can be found all the vertices in Q_n. Take one independent set to be an OS-set, with the i^{th} vertex having a corresponding vector of $\mathbf{e_i}$. Now consider the other half of the vertices. The vectors corresponding to these vertices must be orthogonal to all the other vertices independent of it, and $2^{n-1} - n$ of the vertices in the other independent set. However, the span of all these vectors is contained in the $2^{n-1} - 1$ vectors in the space spanned by the vertices in the same independence set as the vertex in question. So, the span of all those vectors is strictly less than 2^{n-1}, so there exists a linearly independent nonzero vector, which can be found to be orthogonal to the span of the other vectors in that independence set. So, a vector representation exists in $\mathbb{C}^{2^{n-1}}$. □

6.1 Line Graphs

Lemma 6.1. *If* $H \subseteq G$ *then* $L(H)$ *is an induced subgraph of* $L(G)$.

Remark 6.1. $L(C_n) = C_n$ *and* $L(P_n) = P_{n-1}$.

Lemma 6.2. *Let* G *be a graph with vertex* w *and let* $\{v_1, v_2, \ldots, v_k\}$ *be the pendant vertices attached to* w *then if* $G' = G - \{v_1, v_2, \ldots, v_{k-1}\}$ *then*

$$msr(L(G)) = msr(L(G'))$$

Proof. This is a direct result of Theorem [REF] as the edges $v_i w$ in G are duplicates vertices in $L(G)$. $\square$

Theorem 6.8. *If G is a tree then $msr(L(G)) = k = |G| - l$ where k is the number of nonpendant vertices in G and l is the number of pendant vertices in G.*

Proof. If G is a star then $L(G) = K_n$ and $msr(L(G)) = 1$. Induct on the number of nonpendant vertices in G. Assume G has k nonpendant vertices. Let w be a nonpendant vertex where $N(w) = \{v_1, v_2, \ldots, v_k\}$ and $deg(v_i) = 1$ for $i < k$. Let $G' = G - \{v_1, \ldots, v_{k-2}\}$ then $msr(L(G)) = msr(L(G'))$ by Lemma 6.2. In G', $deg(w) = 2$ and $deg(v_{k-1}) = 1$. Then in $L(G')$ the vertex, u, formed from the edge $v_{k-1}w$ has degree one. So $msr(L(G')) = msr(L(G') - u) + 1$. Furthermore the graph $L(G') - u = L(G'')$ where $G'' = G - \{v_1, \ldots, v_k\}$ and it is clear that G'' has $k-1$ nonpendant vertices thus $msr(L(G') - u) = msr(L(G'')) = k - 1$ which directly implies that $msr(L(G)) = k$. $\square$

Remark 6.2. *Theorem 6.8 tells us that the difference between $msr(G)$ and $msr(L(G))$ can be arbitrarily large. Consider the binary tree on $2^n - 1$ vertices then $msr(G) = 2^n - 2$ by Theorem 6.8 $msr(L(G)) = 2^{n-1} - 1$ then $msr(G) - msr(L(G)) = 2^{n-1} - 1$.*

Theorem 6.9. *$msr(L(K_n)) = n - 2$*

This proof is identical to the one found in [REF].

Proof. For $n = 2$, $L(K_2) = K_1$ and $msr(K_2) = 0 = n - 2$. For $n = 3$, $L(K_3) = K_3$ and $msr(K_3) = 1 = n - 2$. For $n = 4$, $L(K_4) = K_{2,2,2}$ and $msr(K_{2,2,2}) = 2 = n - 2$ (???).

So now assume $n \geq 5$. The vertices of $L(K_n)$ will be the unordered pairs from $\{1, \ldots, n\}$. The subgraph induced by a neighborhood of a vertex in $L(K_n)$ is isomorphic to $K_{n-2} \square P_2$, which has minimum rank $n-2$ by Theorem 6.2. Thus $msr(L(K_n)) \geq n-2$.

For the upper bound, let D denote the incidence matrix of an orientation of K_{n-1}. Then $rank(D) = n - 2$. Consider the matrix

$$\begin{bmatrix} I_{n-1} - \frac{1}{n-1}J_{n-1} & D \\ D^T & D^T D \end{bmatrix} = \begin{bmatrix} I_{n-1} - \frac{1}{n-1}J_{n-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ D^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D^T D \end{bmatrix}$$

M is positive semi-definite as postive semi-definite matrices are closed under addition and each term on the rignt is positive semi-definite. Furthermore the martix partition corresponds to the pairs (edges) that contain 1, and those that do not; it is straightforward to check that $M \in P(K_n)$. Since $D^T J_{n-1} = 0$,

$$\begin{bmatrix} I & 0 \\ -D^T & I \end{bmatrix} \begin{bmatrix} I_{n-1} - \frac{1}{n-1}J_{n-1} & D \\ D^T & D^T D \end{bmatrix} = \begin{bmatrix} I_{n-1} - \frac{1}{n-1}J_{n-1} & D \\ 0 & 0 \end{bmatrix}.$$

Since all the columns of $I_{n-1} - \frac{1}{n-1}J_{n-1}$ and of D are orthogonal to the all 1s vector,

$$rank(M) = rank(\begin{bmatrix} I_{n-1} - \frac{1}{n-1}J_{n-1} & D \end{bmatrix}) \leq n - 2$$

so $msr(L(K_n)) \leq n - 2$. $\square$

Corollary 6.4. $msr(L(G)) \leq n - 2$

Proof. If $|G| = n$ then $G \subseteq K_n$, then by Lemma 6.1 $L(G)$ is an induced subgraph of $L(K_n)$ which implies that $msr(L(G)) \leq msr(K_n)$. ☐

Corollary 6.5. *If G has $n \geq 2$ vertices and contains a Hamiltonian path, then $msr(L(G)) = n - 2$.*

Proof. As G contains a Hamiltonian path $P_n \subseteq G$. Then by Lemma 6.1 $L(P_n) = P_{n-1}$ is an induced subgraph of $L(G)$ thus $n - 2 = msr(P_{n-1}) \leq msr(L(G))$. ☐

Theorem 6.10. $msr(K_s \square K_t) = s + t - 2$

Proof. $K_k \square K_{n-k}$ is isomorphic to $L(K_{k,n-k})$ which implies that $msr(K_k \square K_{n-k}) = msr(L(K_{k,n-k})) \leq n - 2$ which directly implies that $msr(K_s \square K_t) \leq s + t - 2$. To establish a lower bound we find a OS-set of size $s + t - 2$. Let $\{v_{i,j}\}$ $1 \leq i \leq s$ and $1 \leq j \leq t$ be the vertices of $K_s \square K_t$ where $\{v_{i,j}\}_{j=1}^{t}$ is a copy of K_t and $\{v_{i,j}\}_{i=1}^{s}$ is a copy of K_s. Then $\{v_{i,1}\}_{i=1}^{s} \cup \{v_{1,3}, v_{1,4}, \ldots, v_{1,t}\}$ is a valid OS set of size $s + t - 2$ which directly implies that $msr(K_s \square K_t) \geq s + t - 2$. ☐

Corollary 6.6. *If $K_{n,n-k} \subseteq G$ then $msr(L(G)) = n - 2$.*

Proof. From the above theorem it is clear that $L(K_{n,n-k}) = n - 2$. ☐

7 Future Work

The definition of the OS-Set given in Section 6 is complicated. One of the goals for this summer was to find an equivalent definition using an algorithm. Although the algorithm produces a valid OS-Set, in general it does produce one of maximum size. Despite the algorithms failure it is hoped that similar approaches may offer insight into the OS-Set.

7.1 OS-set Algorithm

Definition 7.1. *Let G_k be the subgraph induced by vertices $v_1, \ldots, v_k$ as selected by the algorithm.*

Definition 7.2. *Let H_k be the connected component of G_k containing v_k.*

Definition 7.3. *The* partial degree *of vertex v, denoted $pd(v)$, is defined as $deg(v) - k$ where k is the cardinality of $N[H_k] \cap N(v)$.*

Definition 7.4. *The* sum of partial degrees *of vertex v, denoted $\sum pd(v)$, is defined as $\sum pd(u), u \in N(v)$.*

Definition 7.5. *The* reduced sum of partial degrees *of vertex v, denoted $\sum pd_r(v)$, is defined as $\sum pd(u), u \in N(v) \setminus N[H_k]$.*

Algorithm 7.1. ***Input:*** *A simple graph G with no duplicate vertices and no degree two non-simplicial vertices.*

Output: *A Set $S = \{v_1, v_2, \ldots, v_k\}$.*
Let v_1 be a vertex of minimum degree.
While *$pd(v) \neq 0$* ***for some*** *$v \in V(G)$**:***

1. *To select v_{k+1}, first choose any v with $pd(v) = 0$ where $v \notin N(H_k)$. Otherwise, select a v with $pd(v) > 0$ such that $pd(v)$ is minimized. If there is more than one vertex in G satisfying this condition, select one of these vertices such that $\Sigma pd(v)$ is minimized. If there is more than one vertex in G satisfying this condition, select any one of the vertices for which $\Sigma pd_r(v)$ is minimized.*

It is simple to prove that the set selected is a valid OS-set, but failure to consistently select a maximum *OS*-Set invalidates most of our attempts.

7.2 Conjectures

All of the conjectures we put forward here, unless explicitly stated otherwise, hold true for all graphs G where $|G| \leq 7$.

One of the conjectures we support is in fact a conjecture from a prior year of undergraduate research [5]:

Conjecture 7.1. *$msr(G) = OS(G)$.*

For the following, define a *cycle chain* to be a series of cycles on five or more vertices articulated on one edge per cycle where each point of articulation initally has degree two.

Conjecture 7.2. *If G is a cycle chain, then $msr(\overline{G}) = 3$.*

For the previous conjecture, the lower bound has been obtained, but the upper bound has not yet been confirmed. The lower bound and its proof are as follows:

Theorem 7.1. *If G is a cycle chain, then $msr(\overline{G}) \geq 3$.*

Proof. An OS-set of three can be found for any cycle chain. In G, there must be at least one induced P_3. Select one endpoint as v_1. Select the other endpoint as v_2 and select $w_2 \in N(v_1)/N(v_2)$; the one vertex in this set will be adjacent to v_2 in $\overline{G}$ and not adjacent to v_1. For v_3, select any vertex other than those adjacent to the last vertex in the P_3, which will act as w_3 as it is disjoint from both v_1 and v_2 by definition. As $\{v_1, v_2, v_3\}$ is a valid OS-Set, Theorem 6.1 implies that $msr(\overline{G}) \geq 3$. □

The following conjectures relate more generally to $RMPAC$ graphs, which is a graph class which contains cycle chains.

Conjecture 7.3. *If G is an $RMPAC$ graph, then $msr(G) + msr(\overline{G}) \leq |G| + 2$.*

Conjecture 7.4. *If G is an $RMPAC$ graph, then $OS(G) = |G| - 2$.*

Conjecture 7.5. *If G is a connected graph such that for all $v \in V(G)$ $deg(v) \geq k$, then $msr(G) \leq |G| - k$.*

Conjecture 7.5 is a result given in [9] with an incorrect proof. In all known examples it holds true and it would be a beautiful result.

Conjecture 7.6. *$msr(G) + msr(\overline{G}) \leq |G| + 2$.*

Conjecture 7.6 is the full generalization of Conjecture 7.3. It is known to be true in the case of trees [7]. At this point it seems intractible to prove for graphs in general thus the restriction to $RMPAC$ graphs.

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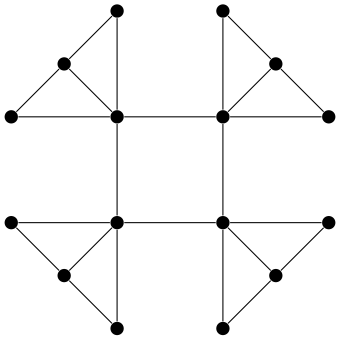

Figure 1: Corona Graph

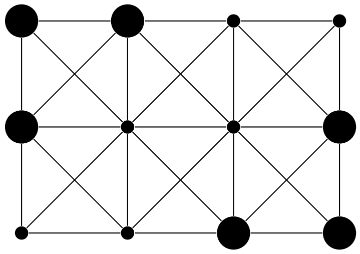

Figure 2: $P_3 \boxtimes P_4$

9 Examples

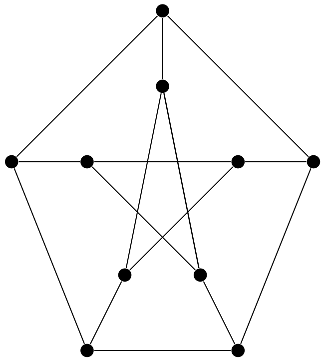

Figure 3: Petersen Graph

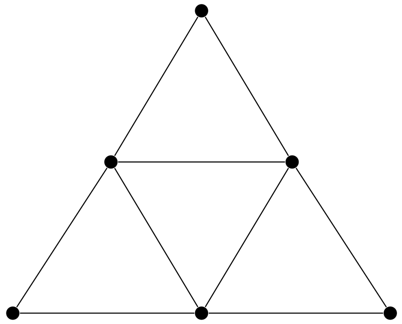

Figure 4: partial 2-tree and MPAC

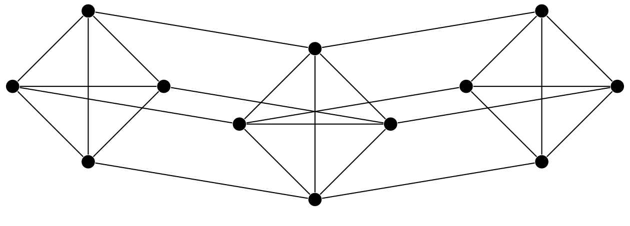

Figure 5: $K_4 \Box P_3$

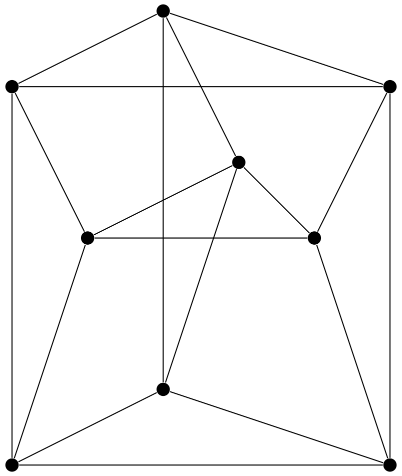

Figure 6: $K_3 \Box K_3$

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