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In Search of a (495, 39, 3) Difference Set

Ken W. Smith

Central Michigan University, Mt. Pleasant, Michigan 44859

It is known that an abelian (495, 39, 3) difference set does not exist. But there are two non-abelian groups of order 495.

This paper reports on a search for a non-abelian (495, 39, 3) difference set. The attack focuses on the contracted difference set which would be created by the 45 cosets of the normal subgroup of order 11. All such contracted difference sets are found. There are exactly two contracted difference sets (up to equivalence) on the non-cyclic group of order 45.

1. An Introduction to Difference Sets

Let $D = \{d_1, d_2, \dots, d_k\}$ be a set of k elements in a group G of order v . D is called a (v, k, λ) - **difference set** in G if for every $x \neq 1, x \in G$, there are exactly λ ordered pairs (d_i, d_j) from D such that $x = d_i d_j^{-1}$.

We refer the reader to the excellent book by Landers, [3], for a detailed investigation into difference sets. Another good text is the book by Hall, [2]. The motivation for studying difference sets is given in the following theorem. (For a proof of the theorem, see [2], pp. 149-50 or [3], pp. 120-1.)

Theorem 0. A difference set in a group G is equivalent to a symmetric (v, k, λ) design with regular automorphism group G .

In this paper we search for a (495, 39, 3) symmetric design by seeking a difference set in one of the four groups of order 495.

The properties of a difference set are unchanged by translation (replacing D with Dg , for some g in G). We may also apply an automorphism of the group without changing the basic properties of the difference set. Therefore, we consider two difference sets D_1 and D_2 in the same group **equivalent** if there exists a group member g and a group automorphism α such that $D_1 = g(D_2)^\alpha$.

The collection of translations and automorphisms of a group G generate a subgroup of $\text{Sym}(G)$. This subgroup is written $\text{Hol}(G)$ and is called the **holomorph** of G . $\text{Hol}(G)$ is the semi-direct product of G and its automorphism group. (See for example [4], p 140.)

From our discussion, $\text{Hol}(G)$ acts on the equivalence classes of difference sets. We may then, in constructing a difference set, choose a suitable translation and then a suitable group automorphism before beginning our search.

We construct an incidence matrix for a difference set D by labeling the rows and columns by group elements and placing a 1 in position (g,h) iff $g \in Dh$ (i.e. gh^{-1} is in D). (Equivalently, we set up a "division" table for the group and replace elements in the table by their image under the characteristic function of D .) Action $(h \rightarrow hx)$ on columns of the incidence matrix corresponds to action on rows $(g \rightarrow gx)$.

The incidence matrix A of the difference set D satisfies the equation

$$AA^t = (k-\lambda)I + \lambda J$$

where I is the identity matrix and J is a matrix of all ones (of appropriate size).

2. Contracted Difference Sets

Let H be a normal subgroup of G of order p . Set $u = v/p$. A difference set D induces on G/H the multi-set $D_H = \{Hg : g \in D\}$. The multiplicity of Hg in D_H is the cardinality of $Hg \cap D$. D_H is called the contracted difference set with parameters $(u=v/p, k, \lambda p)$.

A contracted difference set may be represented by a contracted incidence matrix A_H which has integral entries. We label the rows and columns by group elements and set $(A_H)_{(g,h)} = |Hgh^{-1} \cap D|$. (This is the "division" table approach; the characteristic function of D is now the "multiplicity" function.) A_H is a matrix with nonnegative integral entries which satisfies the equation

$$A_H A_H^t = (k-\lambda)I + \lambda p J.$$

Let the entries of a row of the incidence matrix A_H be a_1, a_2, \dots, a_u . Let m_i be the number of times i occurs in this list. Then

$$\sum_{i=0}^{\infty} m_i = v/p, \quad \sum_{i=0}^{\infty} m_i i = k, \quad \text{and} \quad \sum_{i=0}^{\infty} m_i i^2 = k-\lambda + \lambda p.$$

These three equations place necessary (but not sufficient) conditions on the entries in a row of an incidence matrix of a contracted difference set. We will refer to these equations as the integral-matrix equations.

Notation. A multi-set $\{a_1, a_2, \dots, a_u\}$ will be abbreviated $\{0^{m_0}, 1^{m_1}, 2^{m_2}, \dots, k^{m_k}\}$. We suppress terms of the form i^0 . (For example, we abbreviate the multi-set $\{9, 15, 15\}$ as $\{9, 15^2\}$.)

Most theorems about difference sets carry over to similar statements about contracted difference sets. (The proofs of the theorems often rely on the *integrality* of the incidence matrices.)

We rewrite some theorems about difference sets from the excellent text by Landers ([3]).

Theorem 1. (A modification of Theorem 4.19, page 150 of [3].) Let D be a (v, k, λ) difference set in an abelian group G . Suppose q is a prime such that q^2 divides $k-\lambda$. Suppose H is a proper subgroup with index w and let e be the exponent of (G/H) . Suppose there exists an integer j such that

$$q^j \equiv -1 \pmod{e}.$$

Let a be the smallest nonnegative integer such that $aw \equiv k \pmod{q}$. Then

$$q \leq k-aw.$$

Example. A group of order 495 has a subgroup of order 5. If the group is abelian, that subgroup is normal with index 99 and exponent either 33 or 99. Let q be 2 and a be 1. ($j = 5$ if e is 33; $j = 15$ if e is 99). Then $k-aw$ is negative. Thus there is no abelian $(495, 39, 3)$ difference set.

Theorem 2. (A modification of Theorem 4.30, page 161 in [3].) Let D be a $(v/p, k, \lambda \cdot p)$ contracted difference set in an abelian (quotient) group G/H . Suppose q is a prime that divides v and $k-\lambda$. Let i be the largest integer such that q^i divides v and suppose there exists an integer j such that

$$q^j \equiv -1 \pmod{v/q^i}.$$

Then the Sylow q -subgroup of G/H is not cyclic.

Example. $3^2 \equiv -1 \pmod{5}$. As Landers points out, this prohibits a $(45, 12, 3)$ difference set in the group Z_{45} although there is one in the group $Z_5 \times Z_3 \times Z_3$. In the "contracted" setting, this prohibits a $(495/11, 39, 3 \cdot 11)$ contracted difference set in the non-abelian group $(Z_{11} \rtimes Z_5) \times Z_9$.

3. In search of a (495, 39, 3) difference set

The largest known symmetric design with $\lambda = 3$ has 71 points. One might optimistically assume symmetric designs are "abundant", that there are infinitely many designs with $\lambda=3$ (they are merely well-hidden). Among parameters that come to mind in a search for a symmetric $(v, k, 3)$ design is one which has order 36. This design appears as a sporadic case in a theorem of Cameron's ([1], page 9). We investigate the possibility that such a design exists as a difference set.

There are two group of order 55: the cyclic one, $Z_{11} \times Z_5$, and the metacyclic one, $(Z_{11} \rtimes Z_5) = \langle x, y: x^{11} = x^5 = 1, yxy^{-1} = x^3 \rangle$. There are two groups of order 9. There are four groups of order 495: they are the direct products of the various groups of orders 55 and 9.

(495 is itself an interesting number. A difference set in such a group, if possessing an involution, would give rise to an automorphism group of order 990, the order of "PSL(2,10)". This raises some interesting combinatorial possibilities.)

In investigating a difference set on 495 points, we examine the four groups. From Theorem 1, the two abelian groups cannot have a difference set. In the non-abelian case, we first contract by the unique Sylow 11-subgroup and examine a possible $(495/11, 39, 3 \cdot 11)$ contracted difference set. By Theorem 2, this group has a non cyclic Sylow 3-subgroup. Therefore, we are interested only in the group

$$(Z_{11} \rtimes Z_5) \times Z_3 \times Z_3.$$

We attack this group by contracting with various normal subgroups.

Step 1. Take a normal subgroup of index 3. A contracted $(495/165, 39, 3 \cdot 165)$ difference set has row sum 39 and inner product 495. There are just two such

contracted difference sets:

$$\{17, 11, 11\} \text{ and } \{15, 15, 9\}.$$

Step 2. Contract by a subgroup of index 5. There are five solutions to the integral-matrix equations: they are $\{6^3, 9, 12\}$, $\{5, 6, 8^2, 12\}$, $\{5^2, 9^2, 11\}$, $\{4, 6, 9, 10^2\}$ and $\{3, 9^4\}$. Only one of these is a contracted difference set:

$$3, 9^4 = \{3, 9, 9, 9, 9\}.$$

Step 3. Contract by a subgroup of index 9 to find contracted $(495/55, 39, 3\cdot 55)$ difference sets. A computer search found 28 solutions to the integral-matrix equations:

$$\sum_{i=0}^{\infty} m_i = 9, \quad \sum_{i=0}^{\infty} m_i i = 39, \quad \sum_{i=0}^{\infty} m_i i^2 = 201.$$

We use several techniques to weed out most of these "solutions".

Suppose we contract by a normal subgroup K of G and then by a subgroup G/H of G/K . The result will be the same as contracting by H . (This is the "third isomorphism" theorem of groups, e.g. [4], p. 26.) So if we wish to examine contractions to the quotient group $Z_3 \times Z_3$, we should keep in mind the result, in Step 1, of contracting once again. The technique, that of contracting *twice*, leads to the following result.

Proposition. *A $(495/55, 39, 3\cdot 55)$ difference set in $Z_3 \times Z_3$ has all multiplicities odd.*

Proof. View the members of $Z_3 \times Z_3$ as points of $AG(2,3)$. With this imagery, the cosets of a subgroup of index 3 are lines. The (contracted) difference set assigns weights (multiplicities) to each of the 9 points and the contracted

difference sets in Step 1 arise as the sums of weights across lines of a parallel class in the geometry. Step 1 insists that the sum of weights on any line be odd. Thus each line has 0 or 2 points of even weight. Given a point x , the lines on x partition the remaining eight points into four pairs. Suppose a point x has even weight. Then each of the 4 pairs must have a member of even weight; thus there are 5 points of even weight and 4 of odd weight. The sum of all 9 weights is therefore even. But that sum is 39 and so we have a contradiction.

There are only four solutions of the 28 which have all members odd. They are $\{9, 5^3, 3^5\}$, $\{7^3, 3^6\}$, $\{7^2, 5^3, 3^3, 1\}$, and $\{7, 5^6, 1^2\}$.

The second multi-set, $\{7^3, 3^6\}$, does not give us a $Z_3 \times Z_3$ difference set. (Some line will have two points of weight 7; the lines parallel to this one will have weights 13 and 9. Upon contraction to Z_3 , we would have a $\{17, 13, 9\}$ difference set, contradicting the results in Step 1.) However, the other three results give difference sets.

A contracted difference set is essentially a function (weight) w' from a quotient group G/H to the nonnegative integers which satisfies certain restrictions with respect to the group action. Viewed this way, we may write a difference set in $Z_r \times Z_c$ as entries in an $r \times c$ array; the entry in the (i,j) position is $w(i,j) = |H+(i-j) \cap D|$.

In this form, here are the three $(495/55, 39, 3 \cdot 55)$ contracted $Z_3 \times Z_3$ difference sets:

$$\begin{array}{ccc} 9 & 5 & 3 \\ 5 & 3 & 3 \\ 3 & 3 & 5 \end{array}$$

$$\begin{array}{ccc} 1 & 7 & 3 \\ 7 & 5 & 3 \\ 3 & 5 & 3 \end{array}$$

7 5 5
5 5 1
5 1 5

Remark. $Z_3 \times Z_3$ has four subgroups of order 4. Upon contraction by a subgroup of order 3 we obtain a contracted difference set from Step 1. In each of the three contracted difference sets, above, there are three subgroups which contract to $\{17, 11, 11\}$ and one subgroup which contracts to $\{15, 15, 9\}$. (As arranged above, the latter subgroup is $\langle(2, 1)\rangle = \{(0, 0), (2, 1), (1, 2)\}$.) We will use this result later.

Step 4. We find the $(15 = 495/33, 39, 3 \cdot 33)$ difference sets on $Z_3 \times Z_5$. There are (by computer) 56 solutions to the integral-matrix equations

$$\sum_{i=0}^{\infty} m_i = 15, \quad \sum_{i=0}^{\infty} m_i i = 39, \quad \sum_{i=0}^{\infty} m_i i^2 = 135.$$

These are listed in Table I, at the end of this paper. We attempt to rule out a large number of these.

As suggested before, we write a $Z_3 \times Z_5$ difference set as a 3 by 5 array; the (i, j) entry is $w(i, j)$. Represented this way, contraction by the subgroup $\{(0,0), (1,0), (2,0)\}$ is equivalent to taking column sums. The work in Step 2 proves that column sums are either 3 (once) or 9 (four times). After reducing modulo 3, the array will be:

$$\begin{array}{ccccc} a & b & c & d & e \\ a+v & b+w & c+x & d+y & e+z \\ a-v & b-w & c-x & d-y & e-z \end{array}$$

since column sums are divisible by three.

Now translate the difference set by adding the element $(1,0)$. The array for $D+(1,0)$ is:

$$\begin{array}{ccccc}
 a-v & b-w & c-x & d-y & e-z \\
 a & b & c & d & e \\
 a+v & b+w & c+x & d+y & e+z
 \end{array}$$

These two arrays represent two rows in the contracted incidence matrix. The two rows should have an inner product of $99 \equiv 0 \pmod{3}$. Direct calculation shows that the inner product of these two arrays is, after reduction modulo 3, equal to $-(v^2+w^2+x^2+y^2+z^2)$. As a square in $GF(3)$ is either 0 or 1, the congruence

$$(v^2+w^2+x^2+y^2+z^2) \equiv 0 \pmod{3}$$

implies either two or five elements in the list v, w, x, y, z are zero. Suppose there are only two elements in that list which are zero (modulo 3). We may, without loss of generality, assume that v and w (only) are zero and thus the difference set (modulo 3) looks like:

$$\begin{array}{ccccc}
 a & b & c & d & e \\
 a & b & c+x & d+y & e+z \\
 a & b & c-x & d-y & e-z.
 \end{array}$$

Now we translate by adding $(0, 2)$ to obtain an array for $D+(0,2)$:

$$\begin{array}{ccccc}
 d & e & a & b & c \\
 d+y & e+z & a & b & c+x \\
 d-y & e-z & a & b & c-x,
 \end{array}$$

and again take the inner product with D . After simplifying (modulo 3) we find that the inner product is $2xz$. As before, this inner product must be congruent to 0, and so at least one of x and z is congruent to 0. This is a contradiction.

Therefore $v \equiv w \equiv x \equiv y \equiv z \equiv 0 \pmod{3}$. This forces the $(495/33, 39, 3\cdot33)$ difference set to have the following array:

$$\begin{array}{ccccc}
 a & b & c & d & e \\
 a & b & c & d & e \\
 a & b & c & d & e,
 \end{array}$$

after reduction modulo 3.

Let us translate the difference set so that the first column has sum 3.

Then, without loss of generality, the first column is either {3, 0, 0} or {1, 1, 1}. This rules out a number of sets in Table I, namely the multi-sets numbered 1, 2, 3, 4, 6, 8, 10, 11, 13, 15, 17, 19, 23, 34, and 41.

All other columns of the 3×5 array have sum 9. Thus a 7 in a column forces a column with {7, 1, 1}. This restriction, coupled with the previous one, rules out multi-sets #5 and #7 in Table I.

A 6 in a column (of the 3×5 array) must occur with a 3 and 0. A 5 occurs in a column with two 2s; a 4 occurs in a column with a 1 and another 4. These observations, coupled with the restriction on the first column, rule out all sets except 9, 20, 27, 30, 31, 32, 37, 38, 43, 46, 51, and 54.

We may rule out #20 and #38 by hand; the author has ruled out 27, 30, 32, 43, 46 and 54 with the help of a micro-computer. However, the multi-sets numbered 9, 31, 37 and 51 do indeed lead to contracted (495/33, 39, 3:33) difference sets. These difference sets are equivalent to the following 3×5 arrays:

#9

```
1 3 3 3 7
1 3 3 3 1
1 3 3 3 1
```

#31

```
1 5 2 2 5
1 2 2 2 2
1 2 5 5 2
```

#37

```
3 3 5 2 2
0 3 2 2 2
0 3 2 5 5
```

#51

```
1 1 4 4 1
1 4 4 4 4
1 4 1 1 4
```

(We abuse the language by referring to these arrays associated with contracted difference sets as "difference sets".)

It is interesting, but not very helpful, that the group elements with odd entries in #31, #37 and #51 form the unique (15, 7, 3) difference set - up to equivalence.

Step 5. From this point on, G is the quotient group formed by factoring $(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by the normal subgroup of order 11. We are now prepared to tackle a contracted (45 = 495/11, 39, 3·11) difference set in $G \cong \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

We record a (contracted) difference set on G as follows. G has a subgroup of order 9. Arrange this subgroup as a 3 by 3 square. Do the same to each of the remaining 4 cosets. The members of $G = \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ form five 3×3 squares; the multiplicity $w(i, j, k)$ appears as the (j, k) entry of square i . The array has this form:

```

* * *   * * *   * * *   * * *   * * *
* * *   * * *   * * *   * * *   * * *
* * *   * * *   * * *   * * *   * * *

```

Contracting by a subgroup of index 5 gives the sum of weights within individual 3×3 squares. Contracting by $\{(0,0,0), (0,1,0), (0,2,0)\}$ corresponds to taking column sums. Contracting by $\{(0,0,0), (1,0,0), (2,0,0), (3,0,0), (4,0,0)\}$ corresponds to summing, across the squares, the weights in the (j, k) position of each square. And so on...

There are four subgroups of G of order 3. Fix one and the fifteen cosets of that subgroup give a contracted difference set from Step 4. Upon contracting again by a subgroup of index 3 we should obtain a difference set from Step 1. This process should be equivalent to beginning with the contracted difference set on 45 points and contracting first with a subgroup of index 9 and then with one of index 3. As remarked at the end of Step 3, we should end up with

the difference set $\{9, 15^2\}$ once and the difference set $\{11^2, 15\}$ three times. However, from Step 4, only difference set #37 allows a contraction to $\{9, 15^2\}$. Therefore, difference set #37 occurs exactly once among the four different contractions to 15 points.

What are the other three contracted difference sets on 15 points?

For a group element x , we may solve for the weight (multiplicity) of x in terms of the weights of the four cosets of order 3 (the "lines") on x and the weight of the coset of order 9 (the "plane" or "square") on x . Explicitly, let H_1, H_2, H_3 , and H_4 be the subgroups of G order 3 and let K be the subgroup of order 9. For a subset S of G , let $w(S)$ be the sum of the weights of members of S . Then

$$3w(x) = [w(H_1x) + w(H_2x) + w(H_3x) + w(H_4x) - w(Kx)] \quad (1)$$

Recall that $w(Kx)$ is 3 or 9 (from Step 2) and so the sum

$$w(H_1x) + w(H_2x) + w(H_3x) + w(H_4x)$$

is divisible by 3.

Translate the difference set so that $w(K) = 3$. K has form equivalent to

$$\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \quad \text{or} \quad \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

However, the first case requires that contracting to fifteen points *always* gives a column with a 3 and two zeros; that is, *all* four contracted difference sets on 15 points must be equivalent to #37. This is impossible. The second square agrees with our earlier observation that difference set #37 occurs exactly once. (The other three contractions to difference sets lead to a column with three ones.)

Suppose, without loss of generality, that $H_1 = \{(0, 0, 0), (0, 1, 0), (0, 2, 0)\}$ is the subgroup of G which, upon contraction, gives difference set #37. Then the column sums of the five 3×3 squares in the array for G may be assumed

to be: 3, 0, 0, 3, 3, 3, 5, 2, 2, 2, 2, 5, 2, 2, 5.

Suppose H_2 is a subgroup of G of order 3 and contraction by H_2 gives difference set #51. In addition, suppose $x = (1, 0, 0)$, that is, $w(x)$ is the entry in the upper left of the second square in the array for G . Now, modulo 3,

$$w(H_1 x) + w(H_2 x) + w(H_3 x) + w(H_4 x) \equiv 0 + 1 + w(H_3 x) + w(H_4 x) \equiv 0.$$

Similarly, if $y = (2, 0, 0)$, we have that

$$2 + 1 + w(H_3 x) + w(H_4 x) \equiv 0$$

This forces both H_3 and H_4 to contract to difference set #9. In this case, the last column, $\{7, 1, 1\}$, of difference set #9 must correspond to the second "square", $\{w(1, i, j) : i, j = 0, 1, 2\}$, and so there is some element of this square, say x , such that $w(H_2 x) = w(H_3 x) = w(H_4 x) = 1$. Then $w(x) = (1/3) [3 + 1 + 1 + 1 - 9]$, which is negative!

We have ruled out the possibility of contracting to difference set #51. In a similar way, we can show that if #37 occurs once and #51 not at all, then #31 cannot occur as a difference set either. In other words:

A contracted (495/11, 39, 3·11) difference set in $G = Z_5 \times Z_3 \times Z_3$ has four contractions to a quotient group of order 15. One of these leads to difference set #37, the other three give difference set #9.

Knowledge of the difference sets on 15 points and Equation (1) imply that the contracted difference set on 45 points has the following array.

$$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 & 1 & * & * & * & * & * & * & * & * & * \\ 1 & 0 & 0 & 1 & 1 & 1 & * & * & * & * & * & * & * & * & * \\ 1 & 0 & 0 & 1 & 1 & 1 & * & * & * & * & * & * & * & * & * \end{array}$$

In addition, column sums are 3, 0, 0, 3, 3, 3, 5, 2, 2, 2, 2, 5, 2, 2, 5 and the 3 by 3 squares with asterisks must be equivalent to

$$\begin{array}{ccc} 3 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

We can say more. Contraction by a subgroup of index 5 should force one of the difference sets on 9 points occurring in Step 3. And we may, without loss of generality, assume that one of the unknown squares is precisely equal to the one above. After all of this is figured in, there are only two possible arrays which obey *all* the various contraction requirements.

These two arrays are:

1 0 0	1 1 1	3 2 2	0 0 3	0 2 1
1 0 0	1 1 1	1 0 0	2 0 1	2 0 1
1 0 0	1 1 1	1 0 0	0 2 1	0 0 3

and

1 0 0	1 1 1	3 2 2	0 0 3	0 0 3
1 0 0	1 1 1	1 0 0	2 0 1	0 2 1
1 0 0	1 1 1	1 0 0	0 2 1	2 0 1

Both of these are contracted (495/11, 39, 3-11) difference sets!

If there is a difference set on 495 points, the unique normal subgroup of order 11 gives a contracted difference set equivalent to one of the two sets above.

The last and final stage in the search for a (495, 39, 3) difference set is to examine the two cases, above, and see if they truly allow a difference set on 495 points. This last step may still require considerable work (we have run out of abelian quotient groups). However, the author believes the project is tractable, and hopes to report on the result at a later date.

Table I

Here are 56 solutions to the integral matrix equations for a $Z_3 \times Z_5$ contracted difference set with sum 39 and inner product 33.

- | | |
|---------------------------|-----------------------------|
| 1. $2^{11} 3^3 8$ | 29. $0^2 1^2 3^9 4 6$ |
| 2. $1 2^{11} 4 5 7$ | 30. $0^3 3^{11} 6$ |
| 3. $1^2 2^8 3^3 5 7$ | 31. $1^3 2^8 5^4$ |
| 4. $1^2 2^9 4^3 7$ | 32. $1^4 2^6 4^2 5^3$ |
| 5. $1^3 2^6 3^3 4^2 7$ | 33. $1^5 2^3 3^3 4 5^3$ |
| 6. $0 2^9 3^2 4^2 7$ | 34. $0 1^2 2^6 3^2 4 5^3$ |
| 7. $1^4 2^3 3^6 4 7$ | 35. $1^6 3^6 5^3$ |
| 8. $0 1 2^6 3^5 4 7$ | 36. $0 1^3 2^3 3^5 5^3$ |
| 9. $1^5 3^9 7$ | 37. $0^2 2^6 3^4 5^3$ |
| 10. $0 1^2 2^3 4^8 7$ | 38. $1^5 2^4 4^4 5^2$ |
| 11. $1^2 2^9 3 4 6^2$ | 39. $1^6 2 3^3 4^3 5^2$ |
| 12. $1^3 2^6 3^4 6^2$ | 40. $0 1^3 2^4 3^2 4^3 5^2$ |
| 13. $0 2^9 3^3 6^2$ | 41. $0^2 2^7 3 4^3 5^2$ |
| 14. $1^3 2^7 3^2 5^2 6$ | 42. $0 1^4 2 3^5 4^2 5^2$ |
| 15. $0 2^{10} 3 5^2 6$ | 43. $0^2 1 2^4 3^4 4^2 5^2$ |
| 16. $1^4 2^5 3^2 4^2 5 6$ | 44. $0^2 1^2 2 3^7 4 5^2$ |
| 17. $0 1 2^8 3 4^2 5 6$ | 45. $0^3 2 3^9 5^2$ |
| 18. $1^5 2^2 3^5 4 5 6$ | 46. $1^6 2^2 4^6 5$ |
| 19. $0 1^2 2^5 3^4 4 5 6$ | 47. $0 1^4 2^2 3^2 4^5 5$ |
| 20. $0 1^3 2^2 3^7 5 6$ | 48. $0^2 1 2^5 3 4^5 5$ |
| 21. $0^2 2^5 3^6 5 6$ | 49. $0^2 1^2 2^2 3^4 4^4 5$ |
| 22. $1^5 2^3 3^2 4^4 6$ | 50. $0^3 2^2 3^6 4^3 5$ |
| 23. $0 1^2 2^6 3 4^4 6$ | 51. $1^7 4^8$ |
| 24. $1^6 3^5 4^3 6$ | 52. $0 1^5 3^2 4^7$ |
| 25. $0 1^3 2^3 3^4 4^3 6$ | 53. $0^2 1^2 2^3 3 4^7$ |
| 26. $0^2 2^6 3^3 4^3 6$ | 54. $0^2 1^3 3^4 4^6$ |
| 27. $0 1^4 3^7 4^2 6$ | 55. $0^3 2^3 3^3 4^6$ |
| 28. $0^2 1 2^3 3^6 4^2 6$ | 56. $0^3 1 3^6 4^5$ |

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