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THE REPRESENTATION THEORY OF TACTICAL CONFIGURATIONS

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§1. Introduction

Linear algebra has played an important part in both the theory of groups and in the study of graphs. Linear algebra is applied to the study of groups by investigating the representations of a group (see [7], for example). The adjacency algebra is the starting point for applications of linear algebra to the theory of graphs ([2] and [8]).

The theory of group representations is deeply intertwined with the study of certain semi-simple matrix algebras (group algebras). One may apply these algebras to the study of combinatorial configurations -- even if there is no obvious group at hand. This is "group representation without the group" ([12], [13], [16], [17], [18], and [19]). Specifically, a tactical configuration has a flag algebra which is the homomorphic image of a group algebra.

This paper reviews the theory of flag algebras of a general $(v,b,t+1,s+1)$ tactical configuration, relying heavily on Ott and Liebler's notes ([17] and [13]). The main result (Theorem 4.6) gives an explicit formula for the element f generating the kernel of the homomorphism from the group algebra $\mathbf{C}D_\infty$ to the flag algebra $\mathcal{H}$. $\mathcal{H}$ is the focus of this paper; the basic results are stated without proof. A more detailed treatise on this subject will appear elsewhere.

A combinatorial structure also has an associated graph (the Levi graph) and the linear algebra of graph theory may be applied. There is

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considerable overlap in the linear algebra of the associated graph and that of the group representation.

Our definitions and notation follow Dembowski ([9]). Points are represented by small letters such as "x" and "y"; blocks will be represented by capital letters. If (x,X) is a flag, then x and X are said to be incident and we write xFX (or XFx).

The flag vector space $\mathbf{C}^F$, is the set of all functions from F to the complex field $\mathbf{C}$. $\mathbf{C}^F$ is a $|F|$-dimensional vector space over $\mathbf{C}$ with a distinguished basis labeled by the flags; for an element f of F, we define the distinguished basis element $\underline{f}$ by: $\underline{f}(\underline{h})=1$ if $h = f$, and $\underline{f}(\underline{h})=0$ if $h \neq f$. This vector space is endowed with a natural Euclidean inner product for which the basis $\{\underline{f} : f \in F\}$ is orthonormal. The function π_f represents the orthogonal projection onto the 1-dimensional subspace generated by $\underline{f}$ (thus, for any linear transformation φ, $(\pi_f \varphi)(h)$ represents the "weight" $\varphi(h)$ gives to f).

A walk of length n, from x to y (in a graph) is a sequence of vertices $x = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = y$, such that (x_i, x_{i+1}) is an edge, for all i $(0 \leq i \leq n-1)$. A trail is a walk in which no successive edges are equal. A walk is closed if $x = y$.

§2 The Levi graph and the flag graph.

Given a (v,b,t+1,s+1) tactical configuration, a bipartite graph is created as follows: The vertices are the elements of $P \cup B$, and the edges are the flags F. This graph is called the Levi graph of the design ([8],

[9], and [13]).

Another graph we may associate with a tactical configuration is the flag graph; it has vertices F and two types of edges, τ-edges and σ-edges. We call two flags (x,X) and (y,Y) τ-adjacent if $x=y$; they are σ-adjacent if $X=Y$. We call the flags adjacent if they are either τ- or σ-adjacent. This graph is the line graph of the Levi graph.

The cliques of the flag graph are precisely those sets of flags of the form $\{(x,X): x \text{ fixed}\}$ or $\{(x,X): X \text{ fixed}\}$. These cliques correspond to the points and blocks, respectively.

Every semiregular graph ([8], p. 62) is the Levi graph of a tactical configuration and every Levi graph of a tactical configuration is semiregular. The Levi graph has even girth. For a given tactical configuration the diameter of the Levi graph is equal to, or one more than, the diameter of the flag graph.

The flag graph is the line graph of the Levi graph and so their spectra are related. If N is the $v \times b$ incidence matrix of the tactical configuration then NN^t is a square $v \times v$ matrix and we may study the spectrum of NN^t.

Write the spectrum of NN^t as

$$\begin{pmatrix} 0, & n_0 = (s+1)(t+1), & n_1, & n_2, & n_3, & \dots, & n_k \\ v-(\rho+1), & m_0 & m_1, & m_2, & m_3, & \dots, & m_k \end{pmatrix}$$

for some k, where $\rho+1$ represents the rank of N. The sum of the m_i is $\rho+1$. The trace of NN^t is $|F| = v(t+1) = \sum m_i n_i$. The n_i are positive as they are nonzero eigenvalues of a semi-definite matrix.

The vertices of the Levi graph may be ordered so that the adjacency matrix is

$$L = \begin{bmatrix} 0 & N \\ N^t & 0 \end{bmatrix}.$$

Then $L^2 = NN^t \oplus N^tN$. The spectrum of L^2 is then

$$\begin{pmatrix} 0, & (s+1)(t+1), & n_1, & n_2, & n_3, & \ldots, & n_k \\ b+v-2(\rho+1), & 2m_0 & 2m_1, & 2m_2, & 2m_3, & \ldots, & 2m_k \end{pmatrix}$$

Theorem 2.16 in [8] (p. 62) relates the spectrum of a semiregular graph and its line graph. As a consequence, the spectrum of the flag graph is

$$\begin{pmatrix} -2, & t-1, & s-1, & s+t, & \{\text{roots of } (x-s+1)(x-t+1)-n_i\} \\ v(t+1)-v-b+m_0, & v-(\rho+1), & b-(\rho+1), & m_0, & \{m_i : i=1, 2, \ldots, k\} \end{pmatrix}$$

The flag graph is regular with valency $s+t$ so m_0 is the number of components of the flag graph.

All of the eigenvalues mentioned above are algebraic integers.

§3 The flag algebra $\mathcal{H} = \langle\sigma, \tau\rangle$.

There are several natural relations on the flags F of a tactical configuration. There is the relation $T = \{(f=(x,X),\ g=(y,Y)) : X \neq Y,\ x=y\}$, the collection of τ-edges. There is $S = \{(f=(x,X),\ g=(y,Y)) : X=Y,\ x \neq y\}$, the collection of σ-edges. And there is the diagonal relation $I = \{(f,f) : f \in F\}$.

Define a σ_n-path to be a path of length n such that each succeeding edge alternates from σ to τ and the last edge is a σ-edge. Similarly, a τ_n-path is of length n, with edges of the type $\ldots,\tau,\sigma,\tau$.

We define the operator σ on $\mathbf{C}^F$ by, for $f \in F$,

$$\sigma(\underline{f}) = \sum \underline{h} \quad ((f,h) \in S).$$

Similarly define τ on $\mathbf{C}^F$ by, for $f \in F$,

$$\tau(\underline{f}) = \sum \underline{h} \quad ((f,h) \in T).$$

We extend these definitions to all of $\mathbf{C}^F$ by insisting that they be linear. Hopefully there will be no confusion between the *operator* σ and the *edge type* σ.

The flag algebra $\mathcal{H} = \mathrm{Env}(\sigma,\tau)$ is generated by all linear combinations of products of σ and τ.

Lemma 3.1. $\sigma^2 = (s-1)\sigma + s1$ and $\tau^2 = (t-1)\tau + t1$.

Abbreviate σ_n for $\sigma\tau\sigma\tau\sigma\ldots$ (where n is the number of terms). Similarly, let τ_n stand for $\tau\sigma\tau\sigma\tau\ldots$ From lemma 3.1, both σ and τ are invertible. We extend the subscript to the full set of integers by defining $\sigma_0 = \tau_0 = 1$, and:

$\sigma_{-2k} = (\sigma_{2k})^{-1}$ and $\tau_{-2k} = (\tau_{2k})^{-1}$ (k is a natural number);

$\sigma_{-2k+1} = \sigma_{-2k}\sigma = (\tau_{2k-1})^{-1}$ and $\tau_{-2k+1} = \tau_{-2k}\tau = (\sigma_{2k-1})^{-1}$ ($k \in N$).

We collect formulae implied by lemma 3.1 and the above definitions. Write $X_n = \tau_n + \sigma_n$ and $Y_n = \tau_n - \sigma_n$. Then, for all natural numbers i,

(3.1) $$X_i X_1 = X_{i+1} + \{(t+s-2)/2\}X_i + \{(t-s)/2\}[Y_i - Y_{i-1}] + \{(t+s)/2\}X_{i-1}$$

(3.2) $$Y_i X_1 = -Y_{i+1} + \{(t-s)/2\}X_i + \{(t+s-2)/2\}Y_i + \{(t-s)/2\}X_{i-1} - \{(t+s)/2\}Y_{i-1}$$

and

(3.3) $$Y_i Y_1 = -X_{i+1} + \{(t+s-2)/2\}X_i + \{(t-s)/2\}[Y_i - Y_{i-1}] + \{(t+s)/2\}X_{i-1}$$

(3.4) $X_iY_1 = Y_{i+1} + \{(t-s)/2\}X_i + \{(t+s-2)/2\}Y_i + \{(t-s)/2\}X_{i-1} - \{(t+s)/2\}Y_{i-1}$

Comparing (3.3) and (3.4) to (3.1) and (3.2) motivates the following:

(3.5) $2Y_{i+j} = X_iY_j + (-1)^j Y_iX_j$

and

(3.6) $2X_{i+j} = X_iX_j + (-1)^j Y_iY_j$, for all natural numbers i and j.

Write $[a,b]_j$ for the number of <u>walks</u> of length j from a to b in the Levi graph and write $[a,b]_j^*$ for the number of <u>trails</u> of length j from a to b in the Levi graph. (In Neumaier's paper [15], $[x,X]_{2j+1}$ is denoted by $\alpha_j[x,X]$; $[x,x]_{2j}$ is written $P_j[x,x]$, and $[X,X]_{2j}$ is written $q_j[X,X]$.)

<u>Some remarks about this notation</u>: $[a,a]_j$ will be *twice* the number of closed walks of length j containing the vertex a. For a pair of points (or blocks) x, y, we assume $[x,y]_0 = \delta_{xy}$ (where δ is the Kronecker delta).

<u>Lemma 3.2.</u> Let $f = (x,X)$ and $g = (y,Y)$ be two flags. Then for any nonnegative number i,

a. $\pi_g(Y_{2i+1} - Y_{2i-1})(f) = [x,y]_{2i} - [X,Y]_{2i}$

and

b. $\pi_g(Y_{2i+2} - Y_{2i})(f) = [X,y]_{2i+1} - [x,Y]_{2i+1}$

The above lemma emphasises the combinatorial significance of the terms Y_i in the algebra $\mathcal{H}$, when i is a positive integer. If i is not positive, we may rewrite Y_i in terms of Y_{-i} as follows:

Lemma 3.3. For all integers i,

(a) $Y_{-2i} = (st)^{-1} Y_{2i}$,

and

(b) if $s = t$ then $Y_{-(2i+1)} = -(t)^{-(2i+1)} Y_{2i+1}$.

§4. The irreducible representations of $\mathbf{CD}_\infty$ and $\mathcal{H}$.

Both $\alpha = (2\sigma)/(s+1)-(s-1)/(s+1)$ and $\beta = (2\tau)/(t+1)-(t-1)/(t+1)$ are involutions. This implies $\mathcal{H}$ is a homomorphic image of $\mathbf{CD}_\infty$, the group algebra of the infinite dihedral group $D_\infty = <\alpha, \beta : \alpha^2 = \beta^2 = 1>$. With this natural homomorphism, I will abuse accepted mathematical standards and view α, β, σ, and τ as being in *either* $\mathbf{CD}_\infty$ or $\mathcal{H}$.

The representations of $\mathbf{CD}_\infty$ are known:

Theorem 4.1.

(a) The linear representations of $\mathbf{CD}_\infty$ are

Rep.	λ_1	λ_2	λ_3	λ_4
x	1	-1	1	-1
y	1	-1	-1	1

(b) The irreducible nonlinear representations of $\mathbf{CD}_\infty$ are of degree two, and are parametrized by a real number θ $(0 < \theta < 2\pi)$. Define $n_\theta = (s+1)(t+1)(1+\cos\theta)/2 = (s+1)(t+1)(\cos\theta/2)^2$. Each representation is equivalent to:

$$w_\theta(\sigma) = \begin{bmatrix} s & 0 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad w_\theta(\tau) = \begin{bmatrix} -1 & n_\theta \\ 0 & t \end{bmatrix} .$$

Passing from $\mathbf{CD}_\infty$ to $\mathcal{H}$, some of these representations disappear. In [13] the representations of $\mathcal{H}$ are determined. That result is summarized below.

Theorem 4.2. (Liebler) If the tactical configuration is connected and the incidence matrix has rank $\rho+1$, then the representation λ_1 has multiplicity 1, λ_2 has multiplicity $|F|-b-v+1$, λ_3 has multiplicity $b-1-\rho$, and λ_4 has multiplicity $v-1-\rho$. The sum of the multiplicities of the representations of degree 2 is ρ. Conjugate eigenvalues of $w_\theta(\tau)$ occur to equal multiplicities.

Compare this result to the spectrum of the flag graph in section 2. $(\sigma+\tau)$ is the adjacency matrix of the flag graph and so the n_θ above correspond to the n_i in section 2.

Define f_θ in $\mathcal{H}$ (or $\mathbf{CD}_\infty$) by setting $f_\theta = \tau_2+\sigma_2-(t-1)\sigma-(s-1)\tau + st+1-n_\theta$. Then f_θ annihilates the subspace affording the representation w_θ and acts like $(n_\varphi-n_\theta)I$ on the subspace affording w_φ. It acts as $((s+1)(t+1)-n_\theta)I$ on the subspaces affording λ_1 and λ_2. It acts like $-n_\theta I$ on the subspaces affording the representations λ_3 and λ_4. Notice that $f_\theta = \sigma_2 + (s+t-n_\theta)I + st\sigma_2^{-1} = \tau_2 + (s+t-n_\theta)I + st\tau_2^{-1}$ is symmetric in σ_2 and τ_2. Label the n_θ as $n_1, n_2, \ldots, n_k$ and the corresponding f_θ as $f_1, f_2, \ldots, f_k$. The set $\{f_i : i=1, \ldots, k\}$ is in the center of $\mathcal{H}$. From this we note that $Y_j f_i = Y_{j+2} + (s+t-n_i)Y_j + stY_{j-2}$.

Define $g_2 = (\tau-\sigma) = Y_1$, $g_3 = (\sigma-s)(\tau+1)$, $g_3' = (\tau-t)(\sigma+1)$, and $g_4 = (\sigma\tau-\tau\sigma) = Y_2$. We define a special operator f as follows. If $\dim(\mathcal{H}) = 4k+j$ ($j=2$ or 4) then define $f = g_j(\prod f_i)$. If $\dim(\mathcal{H}) = 4k+3$ and $t > s$, define $f = g_3(\prod f_i)$. If $\dim(\mathcal{H}) = 4k+3$ and $t < s$, define $f = g_3'(\prod f_i)$.

<u>Lemma 4.3.</u>

(a) $f \equiv 0$ in $\mathcal{H}$ (and thus is in the kernel of the natural homomorphism from $\mathbf{C}D_\infty$ to $\mathcal{H}$).

(b) If $\mathcal{H}$ has k nonlinear representations then $g_4(\prod f_i) = 0$ (in $\mathcal{H}$).

<u>Proof.</u> Verify these results on each of the irreducible representations. ‖

<u>Theorem 4.4.</u> $\mathcal{H}$ is isomorphic to $\mathbf{C}D_\infty/\langle f\rangle$.

<u>Proof.</u> The dimension of $\mathcal{H}$ is equal to the dimension of $\mathbf{C}D_\infty/\langle f\rangle$. ‖

Let Γ_k^i represent the symmetric polynomial, all of whose terms are of degree i, in the k variables $\{n_1, n_2, \ldots, n_k\}$. We agree to set Γ_k^i to zero if either $i < 0$ or $i > k$; in addition $\Gamma_k^0 \equiv 1$.

<u>Lemma 4.5.</u> Let $f_i = \tau_2 + (s+t-n_\theta)I + st\tau_2^{-1}$. The coefficient

of Y_{i+2m} $(-k \le m \le k)$ in $Y_i \prod_{\theta=1}^{k} f_\theta$ is $\sum_{j=0}^{k} (st)^{k-m+j} C(k-m+2j, j) \Gamma_k^{m-2j}$.

<u>Proof.</u> The coefficient of $(\tau_2)^m$ in $\prod_{\theta=1}^{k} f_\theta$ is $\sum_{j=0}^{k} (st)^{k-m+j} C(k-m+2j, j) \Gamma_k^{m-2j}$.

Now the f_θ are unchanged by interchanging σ_2 and τ_2 and the f_θ commute with the Y_i. Thus we may multiply $\prod f_\theta$ by $Y_i = \tau_i - \sigma_i$, distribute τ_i and σ_i, and reassemble our answer, choosing τ_2 or σ_2 as the situation requires. ‖

We have, by lemma 3.2, a combinatorial significance to Y_i where i is positive. By lemma 3.3, we may transform Y_{-i} into Y_i in certain situations. If dim($\mathcal{H}$) is 4k+2 or 4k+4, we have $\mathcal{H}$ isomorphic to $CD_\infty/\langle f \rangle$ where f is $Y_1 \prod f_\theta$ or $Y_2 \prod f_\theta$, and so we may calculate f explicitly in terms of the Y_i. The result is summarized below.

<u>Theorem 4.6.</u>

(a) Suppose dim($\mathcal{H}$) = 4k+4. Then $f = Y_{2k+2} + (\Gamma_k^{1}) Y_{2k} +$

$$\sum_{m=0}^{k-2} Y_{2m+2} \left\{ \sum_{j=m}^{\lfloor (k+m)/2 \rfloor} (st)^{j-m} C(2j-m, j) \Gamma_k^{k-2j+m} - \sum_{j=0}^{\lfloor (k+m-2)/2 \rfloor} (st)^{j+1} C(2j+m+2, j) \Gamma_k^{k-2j-m-2} \right\}.$$

(b) Suppose dim($\mathcal{H}$) = 4k+2. Then $f = Y_{2k+1} +$

$$\sum_{m=0}^{k-1} Y_{2m+1} \left\{ \sum_{j=m}^{\lfloor (k+m)/2 \rfloor} (t^2)^{j-m} C(2j-m, j) \Gamma_k^{k-2j+m} - \sum_{j=0}^{\lfloor (k+m-1)/2 \rfloor} (t^2)^{j+1} C(2j+m+1, j) \Gamma_k^{k-2j-m-1} \right\}.$$

Although the equation for f appears somewhat frightening, the equation reduces nicely if k is small. For example, if the dimension of $\mathcal{H}$ is 6 or 8, then respectively,

$$f = Y_3 + (t-n_1)Y_1.$$

$$f = Y_4 + (s+t-n_1)Y_2$$

Using lemma 3.2, we may rewrite these equations in terms of the number of trails between vertices in the Levi graph. Suppose h = (x,X) and g = (y,Y) are flags. Then

(a) if $\dim(\mathcal{H}) = 6$ then $[x,y]^*_2 - [X,Y]^*_2 + (t-n_1+1)(\delta_{xy}-\delta_{XY}) = 0$.

(b) if $\dim(\mathcal{H}) = 8$ then $[y,X]^*_3-[x,Y]^*_3+(s+t-n_1+1)([y,X]^*_1-[x,Y]^*_1)=0$.

These results allow us to classify the tactical configurations with small dimension. In the smallest dimensions ($\dim(\mathcal{H}) < 8$), the restrictions on walks in the flag graph force $[x,y]_2$ (or $[X,Y]_2$) to be constant independent of the pair of points (or blocks) chosen. If $\dim(\mathcal{H}) \geq 8$ then the results are not as compact. (See [3], [4], [5], and [14].) In summary -

<u>Theorem 4.7.</u> Let D = (P, B, F) be a tactical configuration with flag algebra $\mathcal{H} = \mathrm{Env}\{\sigma,\tau\}$. Assume the flag graph is connected. Then

(a) $\mathrm{Dim}(\mathcal{H}) < 6$ implies the design is a trivial design.

(b) $\mathrm{Dim}(\mathcal{H}) = 6$ iff D is a symmetric design.

(c) $\mathrm{Dim}(\mathcal{H}) = 7$ iff D, or the dual of D, is a 2-design which is not symmetric.

(d) $\mathrm{Dim}(\mathcal{H}) = 8$ iff D is a 1-1/2 design.

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