

## PROOF OF THE BARKER ARRAY CONJECTURE

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ABSTRACT. Using only elementary methods, we prove Alquaddoomi and Scholtz's conjecture of 1989, that no  $s \times t$  Barker array having  $s, t > 1$  exists except when  $s = t = 2$ .

### 1. INTRODUCTION

Binary sequences and arrays whose out-of-phase aperiodic autocorrelations are collectively small are particularly useful in digital communication systems, especially synchronisation and radar. The search for such sequences and arrays dates from the 1950s [2], [16] and continues to the present day [7], [9], [13], [14]. We define an  $s \times t$  array to be a two-dimensional array  $(a_{ij})$  of complex-valued elements satisfying

$$a_{ij} = 0 \quad \text{unless } 0 \leq i < s \text{ and } 0 \leq j < t.$$

The array is *binary* if all nonzero elements  $a_{ij}$  take values in  $\{1, -1\}$ . The *aperiodic autocorrelation function* of an  $s \times t$  array  $A = (a_{ij})$  is given by

$$C_A(u, v) = \sum_i \sum_j a_{ij} \overline{a_{i+u, j+v}} \quad \text{for integer } u, v \text{ satisfying } |u| < s \text{ and } |v| < t.$$

We refer to an  $s \times 1$  array as a *sequence of length  $s$* , abbreviating the array  $(a_{i0})$  to  $(a_i)$  and its aperiodic autocorrelation function  $C_A(u, 0)$  to  $C_A(u)$ .

Alquaddoomi and Scholtz [1] defined an  $s \times t$  *Barker array* to be an  $s \times t$  binary array  $A$  for which

$$|C_A(u, v)| \leq 1 \quad \text{for all } (u, v) \neq (0, 0).$$

This generalises the notion of a *Barker sequence* from one dimension (the case  $s = 1$  or  $t = 1$ ) to two dimensions; see [10] and [11] for recent nonexistence results for Barker sequences. The  $2 \times 2$  array  $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$  is a Barker array, but it is conjectured that there are no other sizes for a (truly two-dimensional) Barker array:

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**Conjecture 1.1** (Alquaddoomi and Scholtz 1989 [1]). *If an  $s \times t$  Barker array exists for  $s, t > 1$  then  $s = t = 2$ .*

In this paper we prove Conjecture 1.1 using only elementary methods. We include short proofs of key auxiliary results obtained elsewhere, in order to make the paper self-contained. Theorem 1.2 summarises the previous state of knowledge regarding Conjecture 1.1.

**Theorem 1.2** (Jedwab [6], Jedwab, Lloyd and Mowbray [8]). *Let  $A$  be an  $s \times t$  Barker array with  $s, t > 1$ . Then*

Case 1.  $s, t$  even:  $s = t$ . *If  $t > 2$  then  $t \equiv 0 \pmod{4}$  and  $t \geq 12$ .*

Case 2.  $s$  even,  $t > 1$  odd:  $s > t$ .  $s = 4S^2$  and  $t = T^2$  for integers  $S, T$ . *There exists a Barker sequence of length  $s$ .*

Case 3.  $s, t > 1$  odd:  $st \geq 3^{11}$ . *Write  $t = \prod_j p_j^{\alpha_j}$ , where the  $\{p_j\}$  are distinct primes and  $\alpha_j \geq 1$  for all  $j$ . Then  $\alpha_j \geq 2$  for all  $j$  and  $\alpha_k > 2$  for some  $k$ . If  $st \equiv 1 \pmod{4}$  then  $p_j \equiv 1 \pmod{4}$  for all  $j$ .*

Following [1], define the following function for an  $s \times t$  array  $A = (a_{ij})$ :

$$(1.1) \quad P_A(u, v) = C_A(u, v) + C_A(u, v - t) \quad \text{for } -s < u < s \text{ and } 0 \leq v < t.$$

Any expression involving  $P_A(u, v)$  or  $C_A(u, v)$  will implicitly refer only to values of  $(u, v)$  for which the function is defined. In terms of the array elements  $a_{ij}$  we have

$$(1.2) \quad P_A(u, v) = \sum_i \sum_{j=0}^{t-1} a_{ij} \overline{a_{i+u, (j+v) \bmod t}}.$$

Alquaddoomi and Scholtz [1] established Lemma 1.3 for binary arrays, and then used it to prove Proposition 1.4 for Barker arrays. This generalised the approach taken by Tuyn and Storer in their classical paper [15] on the one-dimensional (sequence) case.

**Lemma 1.3** (Alquaddoomi and Scholtz [1]). *Let  $A$  be an  $s \times t$  binary array. Then*

$$P_A(u, v) \equiv P_A(u, v') \pmod{4} \quad \text{for all } (u, v, v').$$

*Proof.* Let  $u, v, v'$  satisfy  $-s < u < s$  and  $0 \leq v, v' < t$ . From (1.2),  $P_A(u, v)$  is the sum of  $(s - |u|)t$  nonzero terms, of which exactly  $[(s - |u|)t - P_A(u, v)]/2$  are  $-1$  and  $[(s - |u|)t + P_A(u, v)]/2$  are  $+1$ . But from (1.2), the product of these nonzero terms is independent of  $v$ . Therefore

$$(-1)^{[(s-|u|)t-P_A(u,v)]/2}$$

is independent of  $v$ , which implies  $P_A(u, v) \equiv P_A(u, v') \pmod{4}$ .  $\square$

**Proposition 1.4** (Alquaddoomi and Scholtz [1]). *Let  $A$  be an  $s \times t$  Barker array with  $st > 2$ . Then*

Case 1.  $s, t$  even:

$$P_A(u, v) = 0 \quad \text{for } (u, v) \neq (0, 0).$$

Case 2.  $s$  even and  $t$  odd:

$$\begin{aligned} P_{AT}(v, u) &= 0 \quad \text{for } (u, v) \neq (0, 0), \\ P_A(u, v) &= \begin{cases} 0 & \text{for } u \text{ even and } (u, v) \neq (0, 0) \\ k(u) & \text{for } u \text{ odd,} \end{cases} \end{aligned}$$

where  $k(u) = 1$  or  $-1$ .

Case 3.  $s, t$  odd:

$$P_A(u, v) = \begin{cases} k & \text{for } u \text{ even and } (u, v) \neq (0, 0) \\ 0 & \text{for } u \text{ odd,} \end{cases}$$

where  $k = 1$  or  $-1$ .

*Proof.* For all  $u, v$  satisfying  $|u| < s$  and  $|v| < t$ ,  $C_A(u, v)$  is the sum of  $(s - |u|)(t - |v|)$  nonzero terms, each of which is  $\pm 1$ . Therefore  $C_A(u, v) \equiv (s+u)(t+v) \pmod{2}$ . The Barker array property then implies

$$(1.3) \quad C_A(u, v) = \pm((s+u)(t+v) \pmod{2}) \text{ for } (u, v) \neq (0, 0).$$

*Case 1.  $s, t$  even:* From (1.3) we have

$$C_A(u, v) = 0 \text{ for } u \text{ or } v \text{ even and } (u, v) \neq (0, 0).$$

Then by (1.1),

$$P_A(u, v) = 0 \text{ for } u \text{ or } v \text{ even and } (u, v) \neq (0, 0).$$

Lemma 1.3 then implies that

$$P_A(u, v) = 0 \text{ for } (u, v) \neq (0, 0).$$

*Case 2.  $s$  even,  $t$  odd:* From (1.3) we have

$$(1.4) \quad C_A(u, v) = \pm((u(1+v) \pmod{2}) \text{ for } (u, v) \neq (0, 0).$$

It follows from (1.1) that

$$P_A(u, v) = \begin{cases} 0 & \text{for } u \text{ even and } (u, v) \neq (0, 0) \\ \pm 1 & \text{for } u \text{ odd.} \end{cases}$$

Lemma 1.3 then implies that

$$(1.5) \quad P_A(u, v) = \begin{cases} 0 & \text{for } u \text{ even and } (u, v) \neq (0, 0) \\ k(u) & \text{for } u \text{ odd,} \end{cases}$$

where  $k(u) = 1$  or  $-1$ , as required.

We next consider the function

$$(1.6) \quad P_{A^T}(v, u) = C_A(u, v) + C_A(u - s, v).$$

From (1.4),

$$(1.7) \quad P_{A^T}(v, u) = 0 \text{ for } u \text{ even and } (u, v) \neq (0, 0).$$

Lemma 1.3 applied to  $A^T$  states that

$$P_{A^T}(v, u) \equiv P_{A^T}(v, u') \pmod{4} \text{ for all } (u, u', v),$$

giving

$$(1.8) \quad P_{A^T}(v, u) = 0 \text{ for } (u, v) \neq (0, 0), \text{ except when } s = 2 \text{ and } (u, v) = (1, 0)$$

(since, when  $s = 2$  and  $v = 0$ , there is no value of  $u$  satisfying the conditions of (1.7)).

To complete the proof of Case 2, we now derive a contradiction for the case  $s = 2$ , so that (1.8) holds without the exception. By assumption  $st > 2$  and  $s = 2$ , so  $t > 1$  and we can choose an even value of  $v$  satisfying  $0 < v < t$ . From (1.5),

$$k(1) = P_A(1, v) = P_A(1, t - v)$$

and so from (1.1) and (1.4),

$$(1.9) \quad \pm 1 = C_A(1, v) = C_A(1, -v).$$

But by (1.8),  $P_{A^T}(v, 1) = 0$  and so from (1.6) we get

$$\begin{aligned} 0 &= C_A(1, v) + C_A(-1, v) \\ &= C_A(1, v) + C_A(1, -v) \end{aligned}$$

since  $C_A(u, v) = C_A(-u, -v)$  for all  $u, v$ . This contradicts (1.9).

*Case 3.  $s, t$  odd:* From (1.3) we have

$$C_A(u, v) = \pm(((1+u)(1+v)) \bmod 2) \text{ for } (u, v) \neq (0, 0).$$

Then by (1.1),

$$P_A(u, v) = \begin{cases} \pm 1 & \text{for } u \text{ even and } (u, v) \neq (0, 0) \\ 0 & \text{for } u \text{ odd.} \end{cases}$$

Lemma 1.3 then implies that

$$P_A(u, v) = \begin{cases} k(u) & \text{for } u \text{ even and } (u, v) \neq (0, 0) \\ 0 & \text{for } u \text{ odd,} \end{cases}$$

where  $k(u) = 1$  or  $-1$ . By symmetry in  $s$  and  $t$  we also obtain

$$P_{A^T}(v, u) = \begin{cases} k'(v) & \text{for } v \text{ even and } (u, v) \neq (0, 0) \\ 0 & \text{for } v \text{ odd,} \end{cases}$$

where  $k'(v) = 1$  or  $-1$ . But, for  $u, v$  even and  $(u, v) \neq (0, 0)$ , by (1.3) the single nonzero contribution to  $P_A(u, v) = C_A(u, v) + C_A(u, v - t)$  and to  $P_{A^T}(v, u) = C_A(u, v) + C_A(u - s, v)$  is the same term  $C(u, v)$ , and so  $k(u) = k'(v) = k$ .

□

Proposition 1.4 is implied by Theorem 2 and equations (21)–(23) of [1]. Lemma 3.5 of [6] shows that an  $s \times t$  binary array  $A$  having  $P_A(u, v) = 0$  for all  $(u, v) \neq (0, 0)$  is equivalent to  $A$  being simultaneously a perfect binary array and a “quasiperfect” binary array. This in turn is equivalent to the  $-1$  elements of  $A$  corresponding to a  $(4N^2, 2N^2 - N, N^2 - N)$ -difference set in  $\mathbb{Z}_s \times \mathbb{Z}_t$ , where  $st = 4N^2$  (see [4], for example); and the  $-1$  elements of  $\begin{bmatrix} A \\ -A \end{bmatrix}$  corresponding to an  $(st, 2, st, st/2)$  relative difference set in  $\mathbb{Z}_{2s} \times \mathbb{Z}_t = \langle x \rangle \times \langle y \rangle$ , where  $x^{2s} = y^t = 1$ , relative to  $\langle x^s \rangle$  (see [17]). See [3] or [12] for background on difference sets and relative difference sets.

## 2. PROOF OF THE CONJECTURE

We begin with two lemmas.

**Lemma 2.1.** *Let  $A = (a_{ij})$  be an  $s \times t$  binary array and let  $\zeta$  be a (not necessarily primitive)  $t^{\text{th}}$  root of unity. Let  $X = (x_i)$  be the complex-valued sequence of length  $s$  given by*

$$(2.1) \quad x_i = \sum_j a_{ij} \zeta^j.$$

Then

$$C_X(u) = \sum_{v=0}^{t-1} P_A(u, v) \zeta^{-v} \quad \text{for all } u.$$

*Proof.* From (1.2), for all  $u$ ,

$$\begin{aligned} \sum_{v=0}^{t-1} P_A(u, v) \zeta^{-v} &= \sum_{v=0}^{t-1} \sum_i \sum_j a_{ij} \overline{a_{i+u, (j+v) \bmod t}} \zeta^{-v} \\ &= \sum_i \sum_j a_{ij} \sum_{k=0}^{t-1} \overline{a_{i+u, k}} \zeta^{j-k}, \end{aligned}$$

writing  $k = (j + v) \bmod t$  and using  $\zeta^t = 1$ . Hence, for all  $u$ ,

$$\begin{aligned} \sum_{v=0}^{t-1} P_A(u, v) \zeta^{-v} &= \sum_i \sum_j a_{ij} \zeta^j \sum_k \overline{a_{i+u, k} \zeta^k} \\ &= \sum_i x_i \overline{x_{i+u}} \\ &= C_X(u), \end{aligned}$$

as required.  $\square$

**Lemma 2.2.** Let  $X = (x_i)$  be a complex-valued sequence of length  $s$  for which

$$C_X(u) = 0 \quad \text{for } u \neq 0.$$

Then, for some  $I$  satisfying  $0 \leq I < s$ ,

$$|x_i|^2 = \begin{cases} 0 & \text{for } i \neq I, \\ C_X(0) & \text{for } i = I. \end{cases}$$

*Proof.* By the definition of aperiodic autocorrelation, we are given that

$$(2.2) \quad \sum_i x_i \overline{x_{i+u}} = 0 \quad \text{for } 0 < u < s.$$

We prove by induction on  $s$  that, for some  $I$  satisfying  $0 \leq I < s$ ,

$$|x_i|^2 = 0 \quad \text{for } i \neq I.$$

The case  $s = 1$  is immediate (take  $I = 0$ ). Assume case  $s - 1$  to be true. Put  $u = s - 1$  in (2.2) to give  $x_0 \overline{x_{s-1}} = 0$ . This implies, without loss of generality, that  $x_{s-1} = 0$ . Then from (2.2) we have

$$\sum_{i=0}^{s-u-2} x_i \overline{x_{i+u}} = 0 \quad \text{for } 0 < u < s - 1.$$

By the inductive hypothesis it follows that, for some  $I$  satisfying  $0 \leq I < s - 1$ ,  $|x_i|^2 = 0$  for  $i \neq I$ . Combining this with  $x_{s-1} = 0$  gives the case  $s$ , completing the induction.

Furthermore, by the definition of aperiodic autocorrelation,  $C_X(0) = \sum_i |x_i|^2$  and so  $C_X(0) = |x_I|^2$ , as required.  $\square$

The case  $\zeta = 1$  of Lemma 2.1 was used as a starting point in [5], [6] and [8] to derive equations in the row sums  $\sum_j a_{ij}$  of an  $s \times t$  Barker array from Proposition 1.4, leading eventually to Theorem 1.2. We will now use the case where  $\zeta$  is a primitive  $t^{\text{th}}$  root of unity to prove Conjecture 1.1.

**Theorem 2.3.** *If an  $s \times t$  Barker array  $A = (a_{ij})$  exists for  $s, t > 1$  then  $s = t = 2$ .*

*Proof.* Let  $\zeta$  be a primitive  $t^{\text{th}}$  root of unity and define  $X = (x_i)$  as in (2.1). We will show that the case  $s, t$  even forces the result  $s = t = 2$ , whereas the case  $s$  even,  $t$  odd and the case  $s, t$  odd both result in a contradiction. These three cases are exhaustive, because the transpose of a Barker array is also a Barker array.

*Case 1.  $s, t$  even:* Proposition 1.4 and Lemma 2.1 together give

$$C_X(u) = \begin{cases} 0 & \text{for } u \neq 0 \\ st & \text{for } u = 0, \end{cases}$$

using  $P_A(0, 0) = C(0, 0) = st$ . Then by Lemma 2.2 there is some  $I$  satisfying  $0 \leq I < s$  for which

$$(2.3) \quad |x_I|^2 = st.$$

But by (2.1),

$$\begin{aligned} |x_I|^2 &= \left| \sum_{j=0}^{t-1} a_{Ij} \zeta^j \right|^2 \\ &\leq \left( \sum_{j=0}^{t-1} |a_{Ij} \zeta^j| \right)^2 \\ &= t^2. \end{aligned}$$

It follows from (2.3) that

$$(2.4) \quad s \leq t, \text{ with equality } \Leftrightarrow \arg(a_{Ij} \zeta^j) \text{ is constant for all } j \text{ satisfying } 0 \leq j < t.$$

Since  $s$  is even, by symmetry in  $s$  and  $t$  (or equivalently by applying the same procedure to  $A^T$ ) we have  $t \leq s$ , forcing equality. Therefore  $s = t$  and, since  $t > 1$ , by (2.4) we have  $t = 2$ .

*Case 2.  $s$  even,  $t > 1$  odd:* By Proposition 1.4, the  $t \times s$  array  $A^T$  satisfies

$$P_{A^T}(v, u) = 0 \text{ for } (u, v) \neq (0, 0).$$

The argument of Case 1 that led to (2.4), when applied to  $A^T$ , gives  $t \leq s$ . Furthermore the expression for  $P_A$  in Proposition 1.4, together with Lemma 2.1, gives

$$\begin{aligned} C_X(u) &= \begin{cases} 0 & \text{for } u \text{ even and } u \neq 0 \\ k(u) \sum_{v=0}^{t-1} \zeta^{-v} & \text{for } u \text{ odd} \\ st & \text{for } u = 0 \end{cases} \\ &= \begin{cases} 0 & \text{for } u \neq 0 \\ st & \text{for } u = 0, \end{cases} \end{aligned}$$

since  $\zeta^{-1}$  is a primitive  $t^{\text{th}}$  root of unity and  $t > 1$ . By Lemma 2.2 we then obtain  $s \leq t$ , by the same argument as in Case 1. Since we already have  $t \leq s$  this implies  $s = t$ , which contradicts the assumption that  $s$  is even and  $t$  is odd.

*Case 3.  $s, t > 1$  odd:* Proposition 1.4 and Lemma 2.1 together give

$$\begin{aligned} C_X(u) &= \begin{cases} k \sum_{v=0}^{t-1} \zeta^{-v} & \text{for } u \text{ even and } u \neq 0 \\ 0 & \text{for } u \text{ odd} \\ st + k \sum_{v=1}^{t-1} \zeta^{-v} & \text{for } u = 0 \end{cases} \\ &= \begin{cases} 0 & \text{for } u \neq 0 \\ st - k & \text{for } u = 0, \end{cases} \end{aligned}$$

where  $k = 1$  or  $-1$ . Then by Lemma 2.2 there is some  $I$  satisfying  $0 \leq I < s$  for which

$$(2.5) \quad |x_I|^2 = st - k.$$

But, as in Case 1,  $|x_I|^2 \leq t^2$  and so

$$st - k \leq t^2.$$

By symmetry in  $s$  and  $t$  we then have

$$(2.6) \quad st - k \leq \min\{s^2, t^2\}.$$

Suppose, for a contradiction, that  $s \neq t$  and without loss of generality that  $s \geq t + 1$ . Then  $st - k \geq t(t + 1) - k > t^2$ , since  $k = 1$  or  $-1$  and  $t > 1$ . This contradicts (2.6), and so  $s = t$ .

Then (2.6) forces  $k = 1$ , and from (2.1) and (2.5) we have

$$(2.7) \quad \left| \sum_{j=0}^{t-1} a_{Ij} \zeta^j \right|^2 = t^2 - 1.$$

Since  $t$  is odd, one of the sets  $\{j : a_{Ij} = 1\}$  and  $\{j : a_{Ij} = -1\}$  contains at most  $(t-1)/2$  elements; without loss of generality, suppose it is the former. This implies that

$$\begin{aligned} \left| \sum_{j=0}^{t-1} a_{Ij} \zeta^j \right|^2 &= \left| \sum_{j=0}^{t-1} a_{Ij} \zeta^j + \sum_{j=0}^{t-1} \zeta^j \right|^2 \\ &= \left| 2 \sum_{j: a_{Ij}=1} \zeta^j \right|^2 \\ &\leq 4 \left( \sum_{j: a_{Ij}=1} |\zeta^j| \right)^2 \\ &\leq 4 \left( \frac{t-1}{2} \right)^2 \\ &< t^2 - 1, \end{aligned}$$

since  $t > 1$ . This contradicts (2.7). □

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