Assignment 10

This assignment is worth forty points (5,10,10,5,5,5). It is due Monday, December 4, 2006, at the beginning of class.

1. Give an example of an operator T on a vector space V where T does *not* have a minimal polynomial.

2. Let $S_3 = \{1, (123), (132), (12), (13), (23)\}$ be the nonabelian group of six permutations on 3 elements. Consider the set $S = \{P_\sigma : \sigma \in S\}$ of six corresponding 3×3 permutation matrices.

   (a) Show that elements of the set S cannot be simultaneously diagonalized.

   (b) Consider the equilateral triangle centered at the origin with vertices $v_1 = (1, 0), v_2 = (-1/2, \sqrt{3}/2)$ and $v_3 = (-1/2, -\sqrt{3}/2)$. View the permutations in S_3 as permuting the vertices of this triangle and therefore as representing linear operators on $\mathbb{R}^2$. Thus there is a set of six 2×2 matrices which is isomorphic, as a group, to S_3.

   (c) Use part (b) to show that there is a basis of $\mathbb{R}^3$ such that, with respect to that basis, each matrix P_σ has the form $A_\sigma \oplus B_\sigma$ where B_σ is a 2×2 matrix.

3. Let $D_4 = \{1, (1234), (13)(24), (1432), (13), (24), (14)(23), (12)(34)\}$ be the nonabelian group of eight permutations of the square with vertices 1,2,3,4. Consider the set $S = \{P_\sigma : \sigma \in S\}$ of eight corresponding 4×4 permutation matrices.

   Show that there is a basis of $\mathbb{R}^4$ such that, with respect to that basis, each matrix P_σ has the form $A_\sigma \oplus B_\sigma \oplus C_\sigma$ where C_σ is a 2×2 matrix. (Hint: since D_4 acts on the vertices of a square, view the elements of D_4 as linear operators – that is, matrices – by placing the square in the plane, centered at the origin, with vertices $v_1 = (1, 0), v_2 = (0, 1), v_3 = (-1, 0)$ and $v_4 = (0, -1)$. The trace and determinant may also be useful.)

4. Let $f(x) = x^6(x^2 + x + 1)^2(x^2 + 1)^3$ and $p(x) = x^2(x^2 + x + 1)(x^2 + 1)^2$.

   (a) Let $F = \mathbb{R}$, the set of real numbers. Use the rational form to find, up to similarity, all matrices with characteristic polynomial $f(x)$ and minimal polynomial $p(x)$.

   (b) Let $F = \mathbb{C}$, the set of complex numbers. Use the Jordan form to find, up to similarity, all matrices with characteristic polynomial $f(x)$ and minimal polynomial $p(x)$.

5. Use the Jordan form to solve problem 5 of assignment 9.

6. Read about the adjacency matrix of a graph (for example, go to Wikipedia's entry:

   http://en.wikipedia.org/wiki/Adjacency_matrix or Mathworld's entry:

   http://mathworld.wolfram.com/AdjacencyMatrix.html .)

   Then write out the adjacency matrix of the 5-cycle and find its eigenvalues. (Use your work from problem 4 of assignment 9!)