Assignment 9

This assignment is worth fifty points. It is due Wednesday, November 22, 2006 before 3 PM. (Turn it in to my office or my department mailbox.)

None of these problems are from chapter seven of our textbook; please do *not* use the theorems from that chapter.

1. (Hoffman & Kunze, page 208: 3.)
   (a) Let $\mathcal{F}$ be a family of commuting 3×3 complex matrices. How many linearly independent matrices can $\mathcal{F}$ contain?
   (b) What about the $n \times n$ case?

2. Let J_n be the $n \times n$ matrix of all ones. (So $J_3 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.)
   (a) Find a basis of eigenvectors for J_n.
   (b) Let I_n be the $n \times n$ identity matrix and s, t scalars. Diagonalize the matrix $sI_n + tJ_n$.
   (c) Compute the determinant of $sI_n + tJ_n$.

3. Let L_n be the $n \times n$ $(0,1)$-matrix with ones on the "anti-diagonal", that is $L_{i,j} = \delta_{i,n+1-i}$.
   (a) Find a basis of eigenvectors for L_n.
   (b) Let $M_n(a,b)$ be a $2n \times 2n$ matrix with a on the diagonal, b on the "anti-diagonal", and zeros elsewhere – that is, the (i,i) entries of M are a, the $(i, n+1-i)$ entries of M are b, and all the rest of the entries are zero.
   (For example, here is $M_3(2,5)$:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 & 5 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 5 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 & 2 & 0 \\ 5 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.)$$

   (c) Diagonalize $M_n(a,b)$. (Note that $M_n(a,b)$ is in the linear algebra generated by L_{2n}.)

4. Let P be the permutation matrix associated with the permutation $(1,2,3,4,5)$.
   (a) Write out P.
   (b) Given a 5×5 matrix A, what is the effect of pre-multiplying A by P? (That is, how does A compare to PA?)
   What is the effect of *post*-multiplying A by P? (How do A and AP compare?)
   (c) Show that the minimal polynomial of P is $x^5 - 1$.
   (d) Let $\zeta := e^{\frac{2\pi i}{5}} = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$. Note that $\zeta^5 = 1$. Show that the vector $\alpha_1 := \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \\ \zeta^4 \end{pmatrix}$ is an eigenvector for P.

(e) Find a basis for $\mathbb{C}^5$ consisting of eigenvectors of P.

(f) The linear algebra generated by P is the set $< P > = \{f(P) : f \in \mathbb{C}[x]\}$. Give a $\mathbb{C}$-basis for this linear algebra.

(g) Show that the set of elements of the linear algebra $< P >$ can be simultaneously diagonalized. Given a general polynomial $f \in \mathbb{C}[x]$, explicitly give the diagonal matrix corresponding to $f(P)$.

5. Let V be a vector space of dimension n, $T \in L(V, V)$. Suppose T is nilpotent of index k. Prove that the nullity of T is at least $\frac{n}{k}$.

   (Hint: I would do this by induction. Assume as an inductive hypothesis that the claim is true for all linear transformations of index $j < k$. Given T nilpotent of index k, define W to be the nullspace of T^{k-1}. Then W is T-invariant (show this!) and so we may define an operator U which is the restriction of T to W. Note that U is nilpotent of index less than k ....)