Assignment 8

Due Monday, November 13, 2006

This assignment is worth 30 points.

1. For each of the three types of elementary row operations, describe the effect of the operation on the determinant, that is, if A is transformed into A' be an ERO, what is the relationship between the determinant of A and the determinant of A'?

   Prove your claims relying *only* on the definition of the determinant function. Do *not* use a deeper result like $\det(AB) = \det(A)\det(B)$.

2. Suppose $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ where both B and D are square matrices. Prove:

   (a) $\det(A) = \det(B)\det(D)$.

   (b) If f is a polynomial then
   $$f(A) = \begin{pmatrix} f(B) & * \\ 0 & f(D) \end{pmatrix}.$$
   (The asterisk, *, denotes some irrelevant matrix C'.)

   (c) Let p_A, p_B and p_D be the minimal polynomials of the matrices A, B and D, respectively. Show that p_A is divisible by both p_B and p_D and therefore p_A is divisible by the least common multiple of p_B and p_D.
   Is $p_A = \text{lcm}(p_B, p_D)$?

3. Let B be a (0,1) $n \times n$ matrix with ones on the subdiagonal, that is $B_{i,j} = \delta_{i,j+1}$ (where δ is the Kronecker delta.)

   (a) Prove that both the characteristic polynomial and the minimal polynomial of B are equal to x^n.

   (b) Prove that if A is an $n \times n$ matrix with minimal polynomial x^n then A is similar to B. (Hint: I would use induction, looking at the action of A on the range of A.)

4. Suppose A and B are $n \times n$ matrices such that $A^2B = A$. Let W be the row space of A.

   (a) Show that W is a subspace of the row space of B.

   (b) Show that AB, restricted to the subspace W, is the identity operator on W. (The matrix (AB) will act on members of W on the right, since members of W are row vectors. So you need to show that $\omega(AB) = \omega$ for all $\omega \in W$.)

   (c) Show that if A and B have the same rank then W is also an invariant subspace for the matrix B (that is, $\omega \in W \implies \omega B \in W$) and so BA, restricted to the subspace W, is also equal to the identity operator on W.

   (d) Conclude that if A and B have the same rank then $B^2A = B$.