Assignment 6

This assignment is due at the beginning of class on Monday, October 23, 2006. It is worth 40 points.

1. Let V be a vector space of dimension m over a field F. A **bilinear** (or 2-linear) form is a function f from $V \oplus V$ into F such that

   (a) $\forall \alpha, \beta_1, \beta_2 \in V, \forall c_1, c_2 \in F,\ f(\alpha, c_1\beta_1 + c_2\beta_2) = c_1 f(\alpha, \beta_1) + c_2 f(\alpha, \beta_2)$, and

   (b) $\forall \alpha_1, \alpha_2, \beta \in V, \forall c_1, c_2 \in F, f(c_1\alpha_1 + c_2\alpha_2, \beta) = c_1 f(\alpha_1, \beta) + c_2 f(\alpha_2, \beta)$.

   An n-linear form is a function with domain $V \oplus V \oplus V \ldots \oplus V = V^n$ (that is, n direct sums of V) and domain F such that f for each $i, 1 \leq i \leq n$, f is linear in the i-th copy of V if the other entries are held fixed. That is, for all i, $1 \leq i \leq n$, and for all $\alpha_1, \alpha_2, \beta_j \in V, c_1, c_2 \in F$,

   $$f(\beta_1, \beta_2, \ldots, \beta_{i-1},\ c_1\alpha_1 + c_2\alpha_2,\ \beta_{j+1}, \ldots, \beta_n) =$$

   $$c_1 f(\beta_1, \beta_2, \ldots, \beta_{i-1},\ \alpha_1,\ \beta_{j+1}, \ldots, \beta_n) + c_2 f(\beta_1, \beta_2, \ldots, \beta_{i-1},\ \alpha_2,\ \beta_{j+1}, \ldots, \beta_n).$$

   If $n = m$, we may think of V^n as the vector space $Mat_n(F)$ and so n-linear forms on an n-dimensional vector space V agree with the definition of an n-linear function on page 142 in Hoffman & Kunze's textbook.

   An n-linear form f is **alternating** if it has the property that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$ whenever two entries, α_i and α_j are equal.

   (a) Describe all 3-linear forms on F^2.

   (b) What is the dimension of the vector space of n-linear functions on $V = F^n$? (Find a basis.)

   (c) Find all 3-linear alternating forms on F^2.

   (d) Find all 3-linear alternating forms on F^3.

2. (Hoffman & Kunze, page 149, problem 6.) Let K be a subfield of the complex numbers. For an $n \times n$ matrix A over K, define

   $$D(A) = A(j_1, k_1)A(j_2, k_2) \cdots A(j_n, k_n).$$

   Prove that D is n-linear if and only if the integers $j_1, \ldots, j_n$ are distinct.

3. Let F be a finite field of order q. How many different ordered bases are there for the vector space F^n?

   How many different (unordered) bases are there?

4. Let A and B be complex $n \times n$ matrices of the same rank. Show that if $A^2B = A$ then $B^2A = B$.

5. Let D, M be the "differentiate" and "multiply by x" operators on $F[x]$. Identify the linear algebra $< D, M >$ generated by all finite sums of products of D and M.

6. The trace function is defined by $Tr(A) = \sum_{i=1}^{n} A_{ii}$. Show that $Tr : Mat_{n\times n} \to F$ is a linear transformation.