Assignment 5

This assignment is due at the beginning of class on Wednesday, October 11, 2006. It is worth 20 points.

1. (Using a "good" basis) Let

$$B := \{1, x, x^2, x^3\}$$

and

$$B' := \{\alpha_1 := (x-1)(x-2)(x-3), \alpha_2 := x(x-2)(x-3), \alpha_3 := x(x-1)(x-3), \alpha_4 := x(x-1)(x-2)\}.$$

(See problem 1 in Assignment 4.) Let V be the complex vector space of polynomials $\mathbb{C}[x]$ of degree 3 or less. Thus both B and B' are bases for V. Let T be a linear operator on V such that

$$T(\alpha_1) = 2\alpha_1;$$

$$T(\alpha_2) = -2\alpha_2;$$

$$T(\alpha_3) = \alpha_3;$$

$$T(\alpha_4) = -\alpha_4.$$

Write out the matrices $([T]_{B'})^{20}$ and $([T]_B)^{20}$.

2. Let V be the vector space in problem 1, with bases B, B'. Find the dual bases for the two bases B and B'.

3. Let V be the vector space in problem 1, with bases B, B' and D the derivative operator. Given $g \in V^*$ define $f \in V^*$ by $f = Dg$. Compute f if g is

   (a) L_0 (where $L_t(p) := p(t)$),

   (b) L_1,

   (c) L_2,

   (d) $g(ax^3 + bx^2 + cx + d) = 2d - a$.

Addendum. The matrix

$$P := \begin{pmatrix} -6 & 0 & 0 & 0 \\ 11 & 6 & 3 & 2 \\ -6 & -5 & -4 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

is a change of basis matrix for problem 1 of this assignment. This matrix has determinant 12. By a theorem we will prove later, $P^{-1} = \frac{1}{\det(P)}\, adj(P)$, where

$$adj(P) = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 \\ -6 & -12 & -24 & -48 \\ 2 & 6 & 18 & 54 \end{pmatrix}$$

is the adjoint matrix of P.