Assignment 4, revised

This assignment is due at the beginning of class on Wednesday, October 4, 2006. It is worth 20 points.

1. Let

$$B := \{1, x, x^2, x^3\}$$

and

$$B' := \{(x-1)(x-2)(x-3), x(x-2)(x-3), x(x-1)(x-3), x(x-1)(x-2)\}.$$

(See 3(c) in Assignment 3.) Let V be the complex vector space of polynomials $\mathbb{C}[x]$ of degree 3 or less. Thus both B and B' are bases for V.

   (a) Find both the coordinate vectors $[\alpha]_B$ and $[\alpha]_{B'}$ where

      i. $\alpha = x^3$,

      ii. $\alpha = x^3 - 3x^2 + 2x$,

      iii. $\alpha = x^3 + x^2 + x + 1$,

   (b) Let P be a change of basis matrix between the bases B and B' (as described on page 91, before Theorem 14.) Find P.

   (c) Verify your choice of P by showing, for each α in part (a), that $[\alpha]_B = P[\alpha]_{B'}$.

   (d) Let D be the derivative function and T the function from V to V defined by $T(f(x)) = D((x-1)f(x))$. Prove that T is a linear operator for V. Then write out the matrices $[T]_B$ and $[T]_{B'}$.

   (e) Confirm Theorem 14 (page 92) by showing that $[T]_{B'} = P^{-1}[T]_B P$.

2. Let V be the vector space in problem 1, D the derivative operator

   (a) Is there an operator M such that $DM - MD = 1$, that is, that $DM - MD$ is the identity operator for $L(V, V)$?

   (b) If the answer to part (a) is yes, then determine the linear algebra $< D, M >$, generated by D and M, is all of $L(V, V)$. (One approach might be to take the "standard basis" for $L(V, V)$ and see if each element can be written as a linear combination of products involving D and M.)

3. Let V be the vector space in problem 1. Let a be a complex number.

   (a) Show that the function $\phi_a : V \to \mathbb{C}$ defined by $\phi_a(f(x)) = f(a)$ is a linear functional.

   (b) Describe the kernel of ϕ_a.

   (c) Find a basis for the dual space V^*.