MTH 525, Assignment 2
A potpourri of vector spaces

This assignment is worth twenty points. It is due at the beginning of class on Monday, September 18.

1. (10 pts.) For each set V below and field F, decide if V is a vector space over F. (If the operations of addition and scalar multiplication are not defined in the problem, please make the most *natural* assumption....)

   For each set which is *not* a vector space, give an example to show a vector space axiom is violated. For each *vector space* V, provide, *if possible*, a basis for V and give the dimension of the vector space V. (If the dimension is infinite, try to provide the exact cardinality of the dimension.)

   (a) $V := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (under standard vector addition and scalar multiplication); $F := \mathbb{R}$.

   (b) $V := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$ $F := \mathbb{R}$.

   (c) $V := \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$ $F := \mathbb{C}$.

   (d) $V := \{(x, y) \in \mathbb{C}^2 : y = x^2\}$ $F := \mathbb{C}$.

   (e) $V := P_3$, the set of complex polynomials of degree 3 or less (under standard addition and scalar multiplication of polynomials); $F := \mathbb{C}$.

   (f) P_3', the set of polynomials over $\mathbb{C}$ of degree *exactly* 3, under standard addition and scalar multiplication of polynomials; $F := \mathbb{C}$.

   (g) $A_3(4)$, the polynomials p(x) of degree three or less with the property that $p(4) = 0$. (For example, $x^2 - 6x + 8 = (x - 2)(x - 4)$ is a member of $A_3(4)$.) $F := \mathbb{C}$.

   (h) $L_{2,0}$, the set of complex vectors (x, y) on the line $y = 2x$ (under the standard vector addition and scalar multiplication); $F := \mathbb{C}$.

   (i) $L_{2,3}$, the set of complex vectors (x, y) on the line $y = 2x + 3$ $F := \mathbb{C}$.

   (j) $V := \mathbb{C}[x]$, the set of all polynomials over $\mathbb{C}$, $F := \mathbb{C}$.

   (k) $V := \mathbb{C}(x)$, the set of all rational functions over $\mathbb{C}$, $F := \mathbb{C}$.

   (l) V is the nullspace of the complex matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $F = \mathbb{C}$.

   (m) $V := Mat_{\mathbb{C}}(3, 3)$, the set of 3×3 real matrices; $F := \mathbb{C}$.

   (n) $V := C^\infty(\mathbb{C})$, the set of functions from the complex numbers to the complex numbers which are infinitely differentiable, that is, have derivatives of all orders. (Examples include many of the functions from calculus: $f(x) = \sin(x), f(x) = \cos(x), f(x) = e^x$, the set of polynomials, ...etc.) $F := \mathbb{C}$.

   (o) V is the set of all continuous functions from $\mathbb{C}$ into $\mathbb{C}$. $F := \mathbb{C}$.

   (p) Let ω be a fixed complex number and define

   $$V := Trig_\omega(1) = \{a\cos(\omega t) + b\sin(\omega t) : a, b \in \mathbb{C}\}.$$

   $F := \mathbb{C}$. (The algebra operations here are the same as those for $C^\infty(\mathbb{R})$, above.)

   (q) S, the set of all doubly infinite sequences of complex numbers, that is

   $$S := \{(\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) : a_i \in \mathbb{C}\}.$$

   ($F := \mathbb{C}$.)

   (r) S_0, a certain subset of S, above. S_0 is the set of all doubly infinite sequences of complex numbers, with the property that *only a finite number* of coordinates are nonzero. ($F := \mathbb{C}$.)

(s) $V := \mathbb{C}$, the set of complex numbers; $F = \mathbb{R}$, the set of reals.

(t) $F := \mathbb{C}(x)$, the set of rational function in x over the complex numbers $\mathbb{C}$. $V := F[t]$, the set of polynomials over F with indeterminate t.

(u) V is a field of order nine, $F := \mathbb{Z}_3$.

(v) $V := \mathbb{Z}_2[x]$, the set of polynomials over $F := \mathbb{Z}_2$.

(w) V is the set of all subsets of natural numbers. Here addition is "exclusive or", $\oplus$. ($S \oplus T := (S \cup T) - (S \cap T)$.) $F := \mathbb{Z}_2$.

(x) $V := \mathbb{Q}(\sqrt[4]{2})$; $F := \mathbb{Q}$, the set of rationals.

(y) V is the set of algebraic numbers; $F := \mathbb{Q}$. (An algebraic number is a complex number which is the zero of a rational polynomial $p(x) \in \mathbb{Q}[x]$. For example, since $\sqrt{3}$ is a zero of $x^2 - 3$, $\sqrt{3}$ is an algebraic number.)

(z) $V := \mathbb{R}$, the set of real numbers; $F = \mathbb{Q}$.

2. (5 points.) Comb through the *vector spaces* which appear in problem 1. For each field F appearing more than once, create a lattice diagram of the vector spaces over F occuring in problem 1. The diagram should show which vector spaces are subspaces of others. If X is a subspace of Y, then put Y above X in your diagram and draw a line connecting them. For $F = \mathbb{R}$ and $F = \mathbb{C}$ this drawing will cover a full sheet of paper. Please try to draw it neatly. Some vector spaces may not be connected to any others.

   (This exercise is trivial if $F = \mathbb{Z}_2, F = \mathbb{Z}_3$, or $F = \mathbb{C}(x)$. You need not do the exercise for those fields.)

3. (5 points.) Prove that the following sets S are independent sets (of functions from $\mathbb{C}$ to $\mathbb{C}$) over the field of complex numbers $\mathbb{C}$. Try to give an efficient proof of independence; take advantage of properties of the functions.

   (a) $S := \{1, x, x^2, x^3, ..., x^n, ...\}$.

   (b) $S := \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3, ..., \sum_{i=0}^{n} x^i, ...\}$.

   (c) Fix n, a natural number. Define $q(x) := \prod_{i=0}^{n-1}(x-i)$ and for $a \in \mathbb{N}, 0 \leq a \leq n-1$, $p_a(x) := \dfrac{q(x)}{(x-a)}$.
   Set $S := \{p_0(x), p_1(x), p_2(x), \ldots, p_{n-1}(x)\}$.

   (d) $\{\sin(x), \sin(2x), \sin(3x), ..., \sin(nx), ...\}$.
   (The shortest proof I see in this problem uses the fact that $\sin(ax)\sin(bx) = \frac{1}{2}(cos(a-b)x + cos(a+b)x)$ and the fact that $\int_0^{2\pi} \cos(ux)\, dx = 0$ if $u \neq 0$.)

   (e) $S := \{e^x, e^{2x}, e^{3x}, \ldots, e^{nx}, \ldots\}$.