Midterm Exam
SAMPLE PROBLEMS
MTH 623, Spring 2007

(Please note that this exam is considerably longer than the real midterm. It is a compilation of problems which I have thought about putting on the midterm. Some of these problems *will* be on the midterm, some will be *significantly* shortened or modified, some will be left off completely. The real midterm will have a similar format but will be much shorter, 50 to 70 percent the length of this one.)

This exam is worth 200 points. (It is possible to get a score higher than 200; any points over 200 will be viewed as extra-credit.)
It should take you $1\frac{1}{2}$ to 2 hours but you have until 9 PM.

Please do your work on paper provided.

Part 1. (25 points.) Define the following terms:

1. group
2. group homomorphism
3. kernel
4. centralizer
5. center
6. normal subgroup
7. orbit
8. stabilizer
9. equivalence relation

Part 2. (Computations, 120 points.)

1. Let α, β, γ be permutations on the set $\mathbb{Z}$ of integers. Set $\alpha := (0, 2, 4, 6, 8)(1, 3, 5, 7, 9)$, $\beta := (3, 4, 9, 6)(2, 5, 8, 7)$. Let γ be defined by $\gamma(x) = x + 1$.
   (a) Compute
      i. $\alpha\beta$
      ii. $\beta\alpha$
      iii. β^2.
      iv. $\alpha\beta\alpha^{-1}$
      v. $\beta\alpha\beta^{-1}$.
      vi. $\beta^2\alpha\beta^{-2}$.
      vii. $\gamma\alpha\gamma^{-1}$.
      viii. $\gamma\beta\gamma^{-1}$
   (b) For each of the 8 permutations in the part (a), give the order of the permutation.

2. Find a permutation α in A_5 such that $(145) = \alpha(123)\alpha^{-1}$.

3. (a) Let $\mathbb{Z}$ be the group of integers under addition. Find the cosets of $< 3 >= 3\mathbb{Z}$ in $\mathbb{Z}$.

(b) Let $G = Z_{30}$, the integers modulo 30, under addition. Let $H := < 5 >$. Find the cosets of H in G.

(c) Let $K := < (1,2)(3,4), (1,3)(2,4) >$. ($K \leq S_4$.) List the elements of the following left cosets of K:

i. $(1,2)K$,

ii. $(1,2,3)K$.

4. For each of the elements $x \in A_5$, below, first describe the conjugacy class of x in A_5, then describe the centralizer subgroup $C_{A_5}(x)$. (Your description, in each case, should, in addition to other things, include a statement about *size*; your description of the conjugacy class should include giving the size of the class; your description of the centralizer subgroup includes giving the size of the subgroup.)

(a) $x = (1,2,3)$.

(b) $x = (1,2)(3,4)$.

(c) $x = (12345)$.

5. In each problem below, you are given two groups, G and H. Either use the fundamental homomorphism theorem to construct an onto homomorphism from G onto H or prove no such construction exists.

(a) $G = (\mathbb{Z}, +), H = (\mathbb{Z}_5, +)$

(b) $G = S_6, H = (\mathbb{Z}_2, +)$.

(c) $G = D_4, H = (\mathbb{Z}_4, +)$.

(d) $G = D_4, H = (\mathbb{Z}_2 \oplus \mathbb{Z}_2, +)$.

(e) $G = S_3, H = (\mathbb{Z}_2, +)$.

(f) $G = S_3, H = (\mathbb{Z}_3, +)$.

(g) $G = S_4, H = S_3$.

(h) $G = S_4, H = (\mathbb{Z}_6, +)$.

6. A circular necklace has four colored beads. Use Burnside's theorem to find the number of different necklaces which can be made with beads of n colors.

(What is your answer if $n = 3$?)

Part 3. (Proofs, 80 points.)

1. Let $f : G \to H$ be a group homomorphism with kernel K. Prove the following statements.

(a) If 1_G is the identity of G then $f(1_G)$ is the identity of H.

(b) If $x \in G$ then $f(x^{-1}) = f(x)^{-1}$.

(c) K is a subgroup of G.

(d) K is a normal subgroup of G.

(e) If $S < G$ then $f(S) < H$.

2. Choose four statements to prove.

   (a) Let K be a normal subgroup of a group G. Then K is the union of conjugacy classes of G.
   (b) Let $n \geq 4$. Every even permutation of S_n is a product of 3-cycles.
   (c) Conjugation preserves cycle structure, that is, the permutations β and $\gamma := \alpha\beta\alpha^{-1}$ have the same cycle structure.
   (d) Suppose H is a subgroup of G and N is a normal subgroup of G. Prove directly (without using the second isomorphism theorem) that $H \cap N$ is a normal subgroup of H. (You may assume that $H \cap N$ is a subgroup of H.)
   (e) Suppose that G acts on the set X. Let $x \in X$ and let x^G be the orbit of x under the action of G. Then there is a one-to-one correspondence between the elements of x^G and cosets of the stabilizer G_x.
   (f) A_n is simple for $n > 6$. (You may assume that A_6 is simple and that the only normal subgroup of A_n containing a 3-cycle is A_n.)

3. Let $f : G \to H$ be a group homomorphism with kernel K. Prove

   (a) K is a subgroup of G.
   (b) K is a normal subgroup of G.

4. Use the Fundamental Homomorphism Theorem to prove the following:

   (a) (The second isomorphism theorem.) Suppose H is a subgroup of G and N is a normal subgroup of G. Then $G/N \cong H/(H \cap N)$.
   (b) (The third isomorphism theorem.) Suppose $K \leq H \leq G$ and K and H are both normal subgroups of G. Then $(G/K)/(H/K) \cong G/H$.
   (c) The group $Inn(G)$ of all inner automorphisms is isomorphic to $G/Z(G)$. ($Z(G)$ is the center of G.)