Quiz 10
Solutions

1. (2 points.) The group, $U(11)$, of units of 11 under multiplication, is generated by the element 2. Give the subgroup lattice of $U(11)$.

   Solution. The group $< 2 >$ is a cyclic subgroup of size ten and has three proper subgroups, the group $< 4 >$ of order five, the group $< 10 >=< -1 >$ of order two and the trivial group $< 1 >$. The subgroup lattice is thus a diamond with $< 2 >$ on top and $< 1 >$ on the bottom.

2. (6 pts.) There are four nonisomorphic groups which are semidirect products, $\mathbb{Z}_{11} \rtimes \mathbb{Z}_{10}$, of a cyclic group of order 11 by a cyclic group of order 10. Find them all.

   Solution. Let G be generated by elements x of order 11 and y of order 10. The semidirect product $< x > \rtimes < y >$ will require that conjugation by y be an automorphism of $< x >$. $Aut(< x >)$ is isomorphic to $U(11)$. To construct the semidirect product we must construct a homomorphism

   $$\theta :< y >\to Aut(< x >) \cong U(11).$$

   The range of θ must be (isomorphic to) one of the subgroups on problem 1.

   (a) We could have θ map all of $< y >$ to the trivial group, so the kernel of θ is $< y >$ and so

   $$G =< x, y : x^{11} = y^{10} = 1; yxy^{-1} = x > .$$

   This is the abelian group isomorphic to C_{110}.

   (b) We could have θ be an injection, so that the kernel of θ is $\{1\}$ and the range is isomorphic to all of $U(11)$. In this case conjugation maps x to x^s where s is a generator of $U(11)$, say $s = 2$. Therefore

   $$G =< x, y : x^{11} = y^{10} = 1; yxy^{-1} = x^2 > .$$

   (c) We could have θ map all of $< y >$ to a subgroup isomorphic to the subgroup of $U(11)$ generated by $< 4 >$. In this case conjugation maps x to x^4 and so

   $$G =< x, y : x^{11} = y^{10} = 1; yxy^{-1} = x^4 > .$$

   The kernel of θ is the group generated by $< y^5 >$; it has order two.

   (d) We could have θ map all of $< y >$ to a group of size 2 and so conjugation maps x to x^{-1}. The kernel of θ is $< y^2 >$ of order 5. Therefore

   $$G =< x, y : x^{11} = y^{10} = 1; yxy^{-1} = x^{-1} > .$$

   This group is isomorphic to the dihedral group D_{55}.

3. (2 pts.) Find the center of each of the four groups you constructed in problem 2, and thus demonstrate that the four groups are not isomorphic.

   Solution. In the first case the group is abelian so the center is all of G. In the remaining three cases the center is the kernel of the homomorphism θ. (Why?) So the center is trivial ($\{1\}$) in the second group; the center is $< y^5 >$ in the third group and $< y^2 >$ in the last group. Since all four groups have centers of different orders, they are not isomorphic.

Comment.

Except for the first group (the abelian one) these groups are solvable but *not* nilpotent.

The Sylow theorem assures us that all groups of order 110 will have a Sylow 11-subgroup which is normal. We have constructed four nonisomorphic groups of order 110. There are six nonisomorphic groups of order 110; there are two more groups which can constructed as a semidirect product of a cyclic group of order 11 and the dihedral group of order 10.