Quiz 9
Solutions

For the first four problems of this quiz G is a group of order $60 = (3)(4)(5)$.

1. What is the size of the Sylow 2-subgroups? How many Sylow 2-subgroups could G have?

   Solution. There are 1, 3, 5, or 15 Sylow 2-subgroups. They all have size 4.

2. Suppose that G has *fifteen* Sylow 2-subgroups and also has a Sylow 5-subgroup which is not normal. Explain why we can conclude that two of the Sylow 2-subgroups must intersect in a group $< x >$ of order 2.

   Solution. If not, then there are $15 \cdot 3 = 45$ nonidentity elements in the Sylow 2-subgroups and $6 \cdot 4 = 24$ nonidentity elements in the Sylow 5-subgroups. But since no nonidentity element can be in both a Sylow 2-subgroup and a Sylow 5-subgroup, there are at least $45+24 = 69$ nonidentity elements in this group of order 60, a contradiction.

3. Suppose G has two Sylow 2-subgroups P_1 and P_2 which intersect in a subgroup $< x >$ of order 2. What are the possible sizes of the normalizer of this subgroup $< x >$?

   Solution. The groups P_1 and P_2, being of order four, are abelian and so $< x >$ is normal in P_1 and in P_2. Thus both P_1 and P_2 are in the normalizer $N_G(< x >)$. By Lagrange's theorem, $N_G(< x >)$ must have order divisible by four and dividing 60. So the possible sizes of $N_G(< x >)$ are 12, 20, and 60.

4. How does your answer to the previous problem change if you are told that G is a simple group?

   Solution. If $N_G(< x >)$ has size 60 then $< x > \trianglelefteq G$ and so G is not simple. So $N_G(< x >)$ cannot have order 60.

   If $N_G(< x >)$ has size 20 then there is a homomorphism from G into S_3 created by mapping each element g to the permutation created on the three left cosets of $N_G(< x >)$ under left multiplication. This provides a homomorphism from G into S_3. What is the kernel of this homomorphism? Any element *not* in $N_G(< x >)$ is *not* in the kernel of this homomorphism so the kernel cannot be all of G. But neither is the kernel trivial since G has size greater than that of S_3. So we have a nontrivial kernel, violating the simplicity of G. Thus $N_G(< x >)$ cannot have order 20.

   The only size left for $N_G(< x >)$ is 12. (It turns out that this is also impossible – but that is a more difficult argument.)

For the remainder of this quiz, you are to assume that G is a simple group of order 60.

5. Explain why G could *not* have just *one* Sylow 2-subgroup.

   Solution. If it did, that group would be normal, violating the simplicity of G.

6. Explain why G could *not* have just *three* Sylow 2-subgroups.

   Solution. If it did, then conjugation of the Sylow 2-subgroups, $\{P_1, P_2, P_3\}$, of G would provide a homomorphism into S_3. Elements in P_1 not in P_2 would move P_2 so the kernel of this homomorphism is *not* all of G. Since the codomain S_3 is smaller than G, this homomorphism is not injective. Therefore the homomorphism has a nontrivial kernel, violating the simplicity of G.

7. Suppose G has exactly *five* Sylow 2-subgroups. Explain why we can conclude that G is isomorphic to A_5?

   Solution. Conjugation of the five Sylow 2-subgroups $\{P_1, P_2, P_3, P_4, P_5\}$ provides a homomorphism from G into S_5. Elements in P_1 not in P_2 would move P_2 so the kernel of this homomorphism is *not* all of G. Since G is simple, the kernel of this homomorphism must then be just the trivial subgroup $\{1\}$ and so this homomorphism is injective. Therefore G is isomorphic to a subgroup of S_5 of order 60. But S_5 has exactly one subgroup of order 60: A_5.