Solutions to Quiz 8

1. Let G be a group of order $80 = (5)(16)$. Suppose G does not have a normal Sylow 5-subgroup. Answer the following questions under this assumption.

   (a) How many Sylow 5-subgroups does G have?

   (b) Let P be a Sylow 5-subgroup of G. What is the size of $N_G(P)$?

   (c) Explain why G does not have a subgroup of order 10.

   (d) Can G have a subgroup of order 20? order 40? Why/why not?

   Solution. If a group of order 60 does not have a normal Sylow 5-subgroup then it has 16 of them. The normalizer of any Sylow subgroup must then have index 16 and so has order 5.

   If H is a subgroup of order 10 then H has a Sylow 5-subgroup P and $P \trianglelefteq H$. But then $H \leq N_G(P)$ contradicting the fact that $N_G(P)$ has order five.

   Similarly, any subgroup H of order 20 or order 40 has, according to the Sylow theorems, exactly 1 Sylow 5-subgroup P and that subgroup is normal in H so $H \leq N_G(P)$.

2. (continuation of the previous problem) Suppose, instead, that G is a group of order 80 which does have a normal Sylow 5-subgroup. Answer the following questions under the assumption that G has a normal Sylow 5-subgroup.

   (a) How many Sylow 5-subgroups does G have?

   (b) Let P be a Sylow 5-subgroup of G. What is the size of $N_G(P)$?

   (c) Explain why G must have a subgroup of order 10.

   (d) Can G have a subgroup of order 20? order 40? Why/why not?

   Solution. If there is a normal Sylow 5-subgroup P then there is only one and its normalizer is all of G.

   Now G/P is a group of order 16 and such a group has subgroups of orders 2, 4 and 8 and so, by **the correspondence theorem**, G has subgroups of order 10, 20, 40.