Quiz 7
Solutions

For this quiz, p is a prime.

1. Write out the Class Equation for a finite group and derive it (that is, *explain* why it is true.)

   Solution. The class equation is

   $$|G| = |Z(G)| + \sum_x [G : C_G(x)]$$

   where the elements x are representative elements for the conjugacy classes of size greater than one.

   This equation is explained by considering the sizes of conjugacy classes of G. The conjugacy classes of G partition the group G; any conjugacy class of size one is a subset of the center so we may write

   $$|G| = |Z(G)| + \sum_x |x^G|$$

   By the orbit-stabilizer lemma, $|x^G| = [G : C_G(x)]$, so we replace the sizes of orbits with indices of the centralizers to obtain the final form of the class equation.

2. Use the class equation to prove that a group of order p^k has a nontrivial center.

   Solution. Let G be a group of order p^k. By Lagrange's theorem, if the index $[G : C_G(x)]$ is greater than one, then it is a power of p. Consider the class equation and reduce it modulo p to write

   $$0 = |Z(G)| + \sum 0 \mod p$$

   and so $|Z(G)|$ is divisible by p. Since $1 \in Z(G)$, we know that $|Z(G)|$ cannot be zero and so must be at least p.

3. Prove that a group of order p^k has a normal subgroup of order p.

   Solution. The center of a group of order p^k is an abelian group of order divisible by p and so the center $Z(G)$ has a subgroup H of order p. Since *any* subgroup of $Z(G)$ will be a normal subgroup of G, this subgroup H meets the requirements of the exercise.

4. Conclude that a group of order p^k has, for *every* order dividing p^k, a normal subgroup of that order.

   Solution. Use induction (on the exponent of p) and the correspondence theorem.

   By the previous exercise, G has a normal subgroup of order p^1. Now suppose by our induction hypothesis that there is a normal subgroup K of order p^j. The group G/K is a p-group and so it has a normal subgroup N/K of order p; thus N will be a normal subgroup of G of order p^{j+1}.