Quiz 6, Solutions

1. Suppose G is a group of order $112 = 16 \cdot 7$.

   (a) Determine the number of Sylow 2-subgroups and Sylow 7-subgroups of G.

   (b) Suppose a Sylow 7-subgroup P is not normal. What is the size of $N_G(P)$?

   Solution.

   (a) There are 1 or 7 Sylow 2-subgroups; there are 1 or 8 Sylow 7-subgroups.

   (b) If a Sylow 7-subgroup P is not normal then there are 8 conjugates of it and so the index $[G : N_G(P)]$ is equal to 8 by the orbit-stabilizer lemma. Thus $N_G(P)$ is a subgroup of order 14.

2. Let G be a finite group and P a Sylow p-group of G. Prove that if g is a group element of order p^j such that $gPg^{-1} = P$ then g is a member of P.

   Solution. Note that $g \in N_G(P)$. By definition, $P \trianglelefteq N_G(P)$ so we consider the factor group $N_G(P)/P$. If $g \notin P$, then gP is a nonidentity element of $N_G(P)/P$ and so the group $< gP >$ has order p^k for some $k, 1 \leq k \leq j$. Thus, by the correspondence theorem, G has a group of order $p^k \cdot |P|$. But this group is a p-group containing P as a proper subgroup, contradicting the maximality of P.