Quiz 5, February 28, 2007
Solutions

1. The following table describes conjugacy classes of A_4. The table has four rows since there are four conjugacy classes. In each row, the first element is a representative, x, of the conjugacy class x^G. The second item is the centralizer subgroup $C_{A_4}(x)$. The third and fourth items are the sizes, respectively, of the conjugacy class of x and the centralizer subgroup of x. The table has been started for you; finish it (on a separate sheet of paper.)

x	$C_G(x)$	$\|x^G\|$	$\|C_G(x)\|$
1	A_4	1	12
(12)(34)			
(123)			

Solution.

x	$C_G(x)$	$\|x^G\|$	$\|C_G(x)\|$
1	A_4	1	12
(12)(34)	$<(12)(34),(13)(24)>$	3	4
(123)	$<(123)>$	4	3
(132)	$<(132)>$	4	3

2. Use your work in the previous problem to find all normal subgroups of A_4.

Solution. Any normal subgroup of A_4 must be the union of conjugacy classes. Looking at problem 1, we see that there are four conjugacy classes and the first class, $1^G = \{1\}$ must be in every subgroup. Now $\{1\} \cup (12)(34)^{A_4}$ has size 4 and *is* a subgroup so it is a normal subgroup. So the union of the first two conjugacy classes works – it is a normal subgroup!

The union of classes 1^{A_4}, $(12)(34)^{A_4}$ and any other conjugacy class will give something of size $1+3+4=8$ so this cannot be a subgroup. Obviously, the union of *all* the conjugacy classes is A_4, which is a normal subgroup.

So far, using the conjugacy classes of 1 and (12)(34) we have found three normal subgroups,

$$\{1\},$$

$$A_4,$$

and

$$K := <(12)(34),(13)(24)> = \{1,(12)(34),(13)(24),(14)(23)\}.$$

(The subgroups $\{1\}$ and A_4 are "trivial" and "nonproper", respectively; the only *interesting* subgroup is K, a noncyclic subgroup of order 4.)

Could there be others? If we leave out the class $(12)(34)^{A_4}$, our only choices are to use the first conjugacy class and one of the last two. But no combination of sizes (1, 4, and 4) will obey Lagrange's theorem, so there are no more normal subgroups to find.

3. Prove that cycle structure is preserved by conjugation, that is, prove that if α and β are permutations then the permutation $\alpha\beta\alpha^{-1}$ has the same cycle structure as the permutation β.

Solution. Suppose that the cycle structure of β is a certain product of various cycles. Pick an arbitrary element i moved by β and let $j := \beta(i)$, so that β has the form

$$\beta = (\cdots)\ \cdots\ (i,\ j, \cdots)\ \cdots\ (\cdots).$$

We want to show that

$$\alpha\beta\alpha^{-1} = (\cdots)\ \cdots\ (\alpha(i),\ \alpha(j), \cdots)\ \cdots\ (\cdots).$$

that is, that $\alpha\beta\alpha^{-1}$ looks just like β, except any entry x is replaced by $\alpha(x)$.
This is merely a computation: What does $\alpha\beta\alpha^{-1}$ do to $\alpha(i)$?

$$\alpha\beta\alpha^{-1}(\alpha(i)) = \alpha\beta(i) = \alpha(j).$$

So $\alpha\beta\alpha^{-1}$ sends $\alpha(i)$ to $\alpha(j)$. Thus the cycle structure of $\alpha\beta\alpha^{-1}$ is

$$(\cdots)\ \cdots\ (\alpha(i),\ \alpha(j), \cdots)\ \cdots\ (\cdots).$$

(Note that i is arbitrary so we have proven this statement for any entry in the cycle representation of β.)

4. Find all normal subgroups of S_4.

Solution. Here is the list of conjugacy classes of S_4.

x	$C_G(x)$	$\lvert x^G \rvert$	$\lvert C_G(x) \rvert$
1	S_4	1	24
(12)(34)	$< (12)(34), (13)(24), (12) >$	3	8
(123)	$< (123) >$	8	3
(12)	$< (12), (34) >$	6	4
(1234)	$< (1234) >$	6	4

A normal subgroup will be a union of conjugacy classes so we need to find unions of classes which contain 1, have size dividing 24, and are closed under multiplication. There are not very many ways to add 1 to combinations of 3, 8, 6 and 6 and get a divisor of 24. Indeed, the only ways to do that are: 1, 1+3=4, 1+3+8=12, and 1+3+8+6+6=24. These sums all correspond to a normal subgroup; our solution is

$$\{1\},$$

$$K := < (12)(34), (13)(24) >,$$

$$A_4,$$

and

$$S_4.$$

5. An inner automorphism of G is a function $\iota_a : G \to G$ defined by $\iota_a(g) = aga^{-1}$. You may assume that for all a in G, ι_a is an automorphism. Prove that the group $Inn(G)$ of all inner automorphisms is isomorphic to $G/Z(G)$ (where $Z(G)$ is the center of G.)

Solution. Define a map ϕ from G onto $Inn(G)$ by

$$\phi : x \mapsto \iota_x$$

where ι_x is conjugation by x.

Note that ϕ is clearly an onto function.

Then show that ϕ preserves operations:

$$\iota_x \iota_y(g) = \iota_x(ygy^{-1}) = x(ygy^{-1})x^{-1} = (xy)g(y^{-1}x^{-1}) = \iota_{xy}(g)$$

so

$$\phi(x)\phi(y) = \phi(xy).$$

Then observe that the kernel of ϕ is

$$\{x : \iota_x = 1\} = \{x \in G : \iota_x(g) = g, \forall g \in G\} = \{x \in G : xgx^{-1} = g, \forall g \in G\} = \{x \in G : xg = gx, \forall g \in G\}.$$

But this set is just the center, $Z(G)$, of the group G.

Therefore, by the fundamental homomorphism theorem, $Inn(G) \cong G/Z(G)$.