Quiz 4, Spring 2007
Solutions

(Problems are worth 2, 2, 2, and 4 points, respectively.)

1. Let $f : G \to H$ be a group homomorphism. Prove that $f(1_G)$ is the identity of H.

   Solution. Note that for any $x \in G$ we have

   $$f(x)f(1_G) = f(x \cdot 1_G) = f(x).$$

   Since $f(x)$ has an inverse in H we may premultiply both sides by $f(x)^{-1}$ to obtain $f(1_G) = 1_H$.

2. Let $f : G \to H$ be a group homomorphism. Prove that $f(x^{-1}) = f(x)^{-1}$.

   Solution. Verify that

   $$f(x)f(x^{-1}) = f(xx^{-1}) = f(1_G).$$

   But by the previous problem, this is the identity of H so $f(x)$ and $f(x^{-1})$ are inverses.

3. Let $f : G \to H$ be a group homomorphism. Prove that if the kernel of f is $\{1_G\}$ then f is an injection (that is, a one-to-one function.)

   Solution. Suppose the kernel of f is the trivial subgroup and $f(a) = f(b)$. Then

   $$f(a)f(b)^{-1} = 1_H.$$

   By the previous problem, $f(b)^{-1} = f(b^{-1})$ and so $f(a)f(b^{-1}) = 1_H$ which implies (since f is operation preserving) that

   $$f(ab^{-1}) = 1.$$

   Since the kernel of f is trivial then $ab^{-1} = 1$ so $a = b$. We have proven f is one-to-one.

4. Suppose H is a subgroup of G and N is a normal subgroup of G. Prove directly (without using the second isomorphism theorem) that $H \cap N$ is a normal subgroup of H. (You may assume that $H \cap N$ is a subgroup of H.)

   Solution. Let x be any element of H. We need to show that

   $$x(H \cap N)x^{-1} \subseteq (H \cap N).$$

   A typical element of the left-hand side is xgx^{-1} where $g \in H \cap N$. Since N is normal and $g \in N$ then $xgx^{-1} \in N$. And since $g \in H$, by closure of H, $xgx^{-1} \in H$. Therefore $xgx^{-1} \in H \cap N$.

   Thus $x(H \cap N)x^{-1} \subseteq (H \cap N)$ and so $(H \cap N) \trianglelefteq H$.