

## Assignment 8

This assignment is due on Wednesday, April 18, at the beginning of class. Each problem is five points.

1. Let  $H, K$  be subgroups of a group  $G$ . Show that the size of the set  $HK$  is  $\frac{|H||K|}{|H \cap K|}$ .
2. Let  $G$  be a group of order  $p^2q$  where  $p$  and  $q$  are primes. Suppose  $p^2 > q$  and that  $P$  is a Sylow  $p$ -subgroup which is not normal.
  - (a) Show that  $P$  has a subgroup  $K$  of order  $p$  which is normal.
  - (b) Conclude that  $K \text{ char } G$ .
3. Show that any subgroup of order  $p^2q$  is not simple.
4. Suppose  $G$  has a proper subgroup  $H$  of index  $h$ .
  - (a) Show that there is a homomorphism of  $G$  into  $S_h$ .
  - (b) Conclude that if  $G$  is simple then the order of  $G$  must divide  $h!$ .
5. Show that a group of order 112 has a subgroup of order 14.

### Suggestions, hints for this assignment.

1. The size of the set  $H \times K$  is  $|H||K|$ . Map the ordered pair  $(h, k)$  to the element  $hk$ ; this provides a function from the set  $H \times K$  onto the set  $HK$ . Now find the number of pre-images of an element  $x \in HK$  under this map. . . .
2. (a) Consider two Sylow  $p$ -subgroups  $P_1, P_2$ . Show that  $K := P_1 \cap P_2$  has order  $p$ . Then show that  $N_G(K) \leq P_1$  and  $N_G(K) \leq P_2$ . Finally, show that  $K \trianglelefteq G$ .  
(b) Conclude that  $K$  is the intersection of *all* Sylow  $p$ -subgroups and argue that this property forces  $K$  to be characteristic.
3. The hard work has been done in the previous problem.
4. (a) Show that “left multiplication” provides a group action of  $G$  on the left cosets of  $H$ .  
(b) Look at the kernel of the homomorphism in part (a).
5. There are two cases: the Sylow 7-subgroup is normal or it is not. (You *know* what to do with normal subgroups. . . .)