Assignment 7
Applications of the Sylow theorems, continued

This assignment is due on Monday, April 9, at the beginning of class. Each problem is five points.

1. Suppose a group G has k Sylow-p subgroups. Show that conjugation provides a homomorphism from G into the symmetric group S_k.

2. Suppose G is a group of order 12 which does *not* have a normal Sylow-3 subgroup. Show that $G \cong A_4$.

3. Use the Sylow theorems and problem 1 to show that any group of the following orders is not simple.
   (a) 36
   (b) 48
   (c) 72
   (d) 96
   (e) 108
   (f) 24

4. Let p be a fixed prime and so $\mathbb{Z}_p$ is a field. Let $GL_n(p)$ $(= GL(n,p))$ represent the general linear group of invertible matrices over the field $\mathbb{Z}_p$.
   (a) Find the cardinality of $GL_2(p)$ and $GL_3(p)$.
   (b) Show that the set $T := \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in Z_p \}$ is a subgroup of $GL_3(p)$.
   (c) Show that T is a Sylow-p subgroup of $GL_3(p)$.

5. Let G be a group with an element y. Let $< x >$ be a normal subgroup of G.
   (a) Prove that conjugation by y is an automorphism of $< x >$.
   (b) Suppose $yxy^{-1} = x^s$. Show that for any integer j, $yx^jy^{-1} = x^{js}$.
   (c) Suppose $yxy^{-1} = x^s$. Show that for any integer j, $(y^j)x(y^j)^{-1} = x^{s^j}$.

6. Find all groups of order 20 whose Sylow-2 subgroup is cyclic. Give a presentation for each group.

7. Classify all groups of order 28. Give a presentation for each group.

Comments and hints on the assignment.

1. Straightforward.

2. How many Sylow-3 subgroups must there be? Can you show that the homomorphism in problem one is an injection?

3. For a certain prime p, look at the Sylow-p subgroups and use the homomorphism in problem one. You are hunting for a nontrivial proper normal subgroup so you should think about the kernel of this homomorphism.

4. (part (i.)) Each invertible matrix in $GL_n(p)$ corresponds to a change of basis matrix for the vector space $V = \mathbb{Z}_p^n$. So count the number of different bases.

5. The first part is straightforward. Use simple induction to do the last two parts.

6. *Please do not use GAP.* Find a cyclic normal subgroup $< x >$ and let y be an element *not* in $< x >$. Find the possible values of s in the equation $yxy^{-1} = x^s$.

7. *Please do not use GAP.* This problem is very similar to the previous one. In this problem you must deal with two cases: the Sylow-2 subgroup is either cyclic or it is not.