

**Assignment 6**  
**Applications of the Sylow theorems**

This assignment is due on Monday, April 2, at the beginning of class. Each problem is five points.

Please continue to meet expectations for assignments: good style, well-written, with proper acknowledgements.

1. Explain why an abelian finite group with order divisible by a prime  $p$  can have only one Sylow- $p$  subgroup.
2. Use the Sylow theorems to find the number of Sylow subgroups of groups of the following orders:
  - (a) 12
  - (b) 24
  - (c) 30
  - (d) 36
  - (e) 40
  - (f) 42
  - (g) 45
  - (h) 48
  - (i) 56
  - (j) 60
3. List all the integers  $n$ ,  $2 \leq n \leq 60$ , for which the Sylow theorems immediately *guarantee* a normal Sylow subgroup. (Be sure to identify the appropriate Sylow subgroup. For example, you should point out that if  $n = 6$  then the Sylow-3 subgroup is normal, while if  $n = 15$  then both the Sylow-3 and Sylow-5 subgroups are normal.)
4. Let  $p$  and  $q$  be primes where  $p < q$  and  $p$  does not divide  $q - 1$ . Show that any group of order  $pq$  is a direct product of its Sylow subgroups. Conclude then that such a group is cyclic.
5. Show that the Sylow-2 subgroups of  $S_5$  are isomorphic to  $D_4$  and that the Sylow-2 subgroups of  $S_6$  are isomorphic to  $D_4 \times Z_2$ .
6. Use your knowledge of Sylow-2 subgroups to show that the groups  $S_5$  and  $SL(2, 5)$  are not isomorphic.
7. Use the Sylow theorems to show that any group of the following orders is not simple.
  - (a) 30
  - (b) 56
  - (c) 80
  - (d) 105

**Comments and hints on the assignment.**

1. Use the First Sylow Theorem (Theorem 4.12, page 79), part (i).
2. This involves simple computations, using the first Sylow Theorem (part (ii).) The answer to the first subproblem is: "A group of order 12 has one or three Sylow-2 subgroups and one or four Sylow-3 subgroups."

3. Simple computations. But you should carefully organize your list of integers into some classification by types of factors, e.g. (1) primes and prime powers, (2) a prime times a prime power, ... etc. (*Please, do not* just list all the numbers from 2 to 60 and work on them one by one!)
4. I think I did this (quickly) in class one day... Write out the argument carefully.
5. We did part of this problem on the take-home midterm.
6. See the take-home midterm.
7. Count elements of a certain order and force a single Sylow-p subgroup.