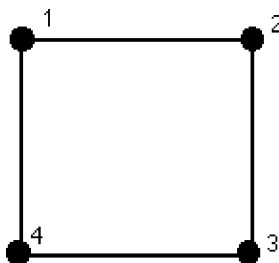


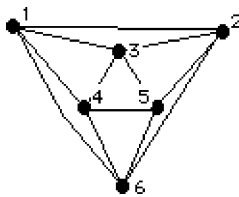
### Assignment 5, Spring 2007

Due at the beginning of class, Monday, February 12

- Let  $S$  be a subset of a group  $G$ . Define a relation  $\sim$  on  $G$  by  $a \sim b \iff a^{-1}b \in S$ .
  - Prove that  $\sim$  is an equivalence relation on  $G$  if and only if  $S$  is a subgroup of  $G$ .
  - Prove that the equivalence classes of  $\sim$  are exactly the left cosets of  $S$ .
  - Prove that the cardinality of the set of right cosets of  $S$  in  $G$  is the cardinality of the set of left cosets of  $S$  in  $G$ . (Warning! We are *not* assuming  $G$  is finite.)
  - Show that Lagrange's theorem follows from the work above.
- Let  $G$  be a simple group and  $\phi : G \rightarrow H$  a group homomorphism. Prove  $\phi$  is either the trivial map or an injection.
- Let  $G$  be the group of all the automorphisms (symmetries) of the following graph:



- List the elements of  $G$ . Describe  $G$ .
  - List the elements of the stabilizer  $G_x$ , for each vertex  $x$ .
  - List the elements of  $G(1 \mapsto 2)$  and show that this set is a coset of  $G_1$ .
  - Explicitly show that  $G_2$  is a conjugate subgroup of  $G_1$ .
4. Let  $G$  be the group of all the automorphisms of the graph:



- List the elements of  $G$ . Describe  $G$ .
  - List the elements of the stabilizer  $G_x$ , for each vertex  $x$ .
  - List the elements of  $G(1 \mapsto 2)$  and show that this set is a coset of  $G_1$ .
  - Show that  $G_2$  is a conjugate subgroup of  $G_1$ .
5. Set  $G := S_{\mathbb{Z}}$ , the symmetric group of permutations of the integers. Let  $H$  be the subgroup of all permutations which only move a finite number of elements. (For any permutation  $\sigma$ , we may define a set  $X_{\sigma} \subseteq \mathbb{Z}$  by  $X_{\sigma} := \{x \in \mathbb{Z} : \sigma(x) \neq x\}$ .  $H$  is then the set of all  $\sigma \in S_{\mathbb{Z}}$  such that  $|X_{\sigma}| < \infty$ .)
- Show that  $H \trianglelefteq G$ .
  - What is the cardinality of  $H$ ?

(c) Show that  $H$  has a subgroup  $A_\infty$  consisting of all members of  $H$  which are *even* permutations.

### Comments on Assignment 5

1. Straightforward. (Recall that two sets have the same cardinality if there is a bijection between them. You found the bijection from  $S$  to  $aS$  in an earlier assignment.)
2. Salivate.
3. It doesn't hurt to think of this graph as a square. . . .
4. Problem 4 (similarly, problem 3) ask you to work out in detail a single example. This is possibly tedious, but offers a chance to develop some intuition about group action.
5. A later assignment will ask you to show that  $A_\infty$  is simple.