Assignment 4, Spring 2007

Due at the beginning of class, Monday, February 5

1. Draw the lattice diagram of the subgroups of G where
   (a) G is the "Klein-Four" group, isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
   (b) $G = S_3$.
   (c) $G = D_4$.
   (d) $G = Q_4$.
   (e) $G = A_4$.

2. Let p and q be distinct primes. Draw the lattice diagram of the subgroups of the following groups.
   (a) $\mathbb{Z}_p$
   (b) $\mathbb{Z}_{p^2}$
   (c) $\mathbb{Z}_{p^n}$, where n is a positive integer.
   (d) $\mathbb{Z}_p \oplus \mathbb{Z}_p$.
   (e) $\mathbb{Z}_{pq}$.

3. (a) Let $G = GL(2, \mathbb{Q})$ and $A, B \in G$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Show that A has order 4, B has order 3, yet AB has infinite order. (Thus we have a solution to the $(3, 4, \infty)$ permutation problem.)
   (b) Show that if $a, b > 1$ then there are solutions to the (a, b, ∞) problem. (You should include the possibility that a or b is ∞.)

4. Let R be a ring with unity and let $Mat_3(R)$ represent the ring of 3×3 matrices with entries from R. Define $m(a, b, c) := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$.
   (a) Find the multiplicative inverse of $m(a, b, c)$ and show that it is a matrix of the same form, that is, $m(a, b, c)^{-1} = m(a', b', c')$ for some $a', b', c' \in R$.
   (b) Show that the product of two matrices of the form $m(a, b, c)$ is again of that form.
   (c) Let S, T be subrings of R. Define the set $M(S, T) := \{m(s, t, r) : s \in S, t \in T, r \in R\}$. Show that $M(S, T)$ is a group under matrix multiplication.
   (d) Identify the commutators of $M(S, T)$.

5. (a) In the previous problem, let $R = S = T = \mathbb{Z}_2$, the field with 2 elements. Then $M(S, T)$ is a nonabelian group of order 8. Which one? (To which of the five groups of order 8 is it isomorphic?)
   (b) In the previous problem, let $R = S = T = \mathbb{Z}_3$, the field with 3 elements. Show that $M(S, T)$ is a nonabelian group of order 27 with the property that $X^3 = 1$ for all X in $M(S, T)$.
   (c) Let G be a group with the following properties:
      i. It has an element x of order 9 and an element y of order 3.
      ii. $yxy^{-1} = x^4$ (and so $yx = x^4 y$.)
      iii. G is generated by x and y.

What is the order of G? (Hint: the subgroup generated by x is normal; first find the order of $G/<x>$.)

(d) Describe, up to isomorphism, *all* groups of order 27.

Comments on Assignment 4

1. No creativity required. This may be slightly tedious, but should provide familiarity with these important groups. (Caution: not every subgroup is cyclic.)

   We will use this knowledge at a later date.

2. Review a basic understanding of cyclic groups. (We will use this knowledge at a later date.)

3. I would do part (b) using permutations on an infinite set. I don't see a way to use matrices. (So don't let part (a) distract you.)

4. All of this is straightforward. (No creativity required.) We will use this knowledge at a later date.

5. You might wish to use GAP to help identify the groups of order 27. Also look at the "Fundamental Theorem of Abelian Groups."

   We will use this knowledge at a later date.