Assignment 2, Spring 2007

Due at the beginning of class, Wednesday, January 24

1. Let F be a field. Define $f, g : F \cup \{\infty\} \to F \cup \{\infty\}$ by $f(x) := 1 - x,\ g(x) := \frac{1}{x}$. Find all the elements of the group generated by f and g under function composition.

   To which well-known group is this group isomorphic?

2. Given subgroup K of group G, list the elements of G/K, the set of left cosets of K in G. What is the **cardinality** of G/K? of K?

   (a) $G = (\mathbb{Z}, +), K =< 10 >$.
   (b) $G = S_3, K =< (12 >$.
   (c) $G = S_3, K =< (123) >$.
   (d) $G = D_4, K =< R_{180} >$.
   (e) $G = D_4, K =< H >$.
   (f) $G = A_4, K =< (12)(34), (13)(24) >$.

3. Let G be a group. Given an element a of G, define the function $L_a : G \to G$ by $L_a(x) = ax$. (We will call this function "left multiplication by a.") Define $R_a : G \to G$ by $R_a(x) = xa$. (This is "right multiplication by a".)

   (a) Show that L_a is a one-to-one function from G onto G (that is, a *bijection* from G to G.)
   (b) Show that for all a, b in G, $L_a L_b = L_{ab}$.
   (c) Show that for all a, b in G, $R_a R_b = R_{ba}$.
   (d) Show that the map $G \to Sym_G$ defined by $a \mapsto L_a$ is an isomorphism from G into Sym_G, the symmetric group of permutations of elements of G.

4. Let G_p be the group of all permutations on the elements of $F = \mathbb{Z}_p$ which can be represented by a permutation polynomial.

   (a) Let H be the subgroup of G_p generated by the permutations which correspond to linear polynomials. Find the order of H. Is H abelian?
   (b) Show that $f(x) = x^3$ is a permutation polynomial for $F = \mathbb{Z}_5$.
   (c) Show that $G_3 \equiv S_3$ and $G_5 \equiv S_5$.
   (d) More generally, can we show that $G_p \equiv S_p$?

5. For which natural numbers (a, b, c) do there exist solutions to the (a, b, c)-permutation problem, described in Assignment 1?

Comments on Assignment 2.

1. Write out a multiplication (Cayley) table for your group, looking at combinations of f, g, fg, etc. There should be less than ten elements.

2. Straightforward (just compute.) We will use this problem in Assignment 3.

3. See p. 54, problem 3.27 for a very similar problem. The problem is all "definition tracing." (It is worth doing – but there is no creativity required.)

4. (c) You might wish to use GAP to generate G_5. (But the problem can be done without GAP.)

   (d) This problem is more challenging. I would begin by looking at a simple polynomial like $f(x) = x^k$ where $GCD(k, p-1) = 1$. You may use, if needed, the result in problem 3.28, on page 54.

5. Challenging.