Assignment 1

Due at 5 PM, Tuesday, January 16, 2007

Problem 4 must be submitted electronically. The other problems may be submitted electronically or on paper at the beginning of class on January 15. (If you submit problems on paper, please put each of the three problems, 1-3, on a separate sheet of paper.)

Your mathematical style and communication are an important part of your answer.

1. (Meditating on permutations) Let's say that the pair of permutations σ and τ "solve the (a, b, c) permutation problem" (or "σ and τ are (a, b, c)-solutions") if the order of σ is a, the order of τ is b and the order of $\sigma\tau$ is c.

   (a) Find σ, τ that are solutions to:

      i. the $(2, 2, 3)$-permutation problem,

      ii. the $(2, 2, 4)$-permutation problem,

      iii. the $(2, 3, 7)$-permutation problem,

      iv. the $(2, 2, \infty)$-permutation problem.

   (b) Show that $(2, 2, n)$-solutions exist for all natural numbers n.

   (c) Show that there are solutions to the $(a, b, LCM(a, b))$-problem for all finite $a, b \geq 2$.

   (d) Show that if there is a solution to the (a, b, c)-problem then there is a solution to the (b, a, c)-problem and the (b, c, a)-problem and therefore a solution to the (x, y, z)-problem where (x, y, z) is any permutation of the triple (a, b, c). (For this reason, we will, in the future, assume $a \leq b \leq c$.)

2. Get the software *GAP* on a computer you can use on a regular basis. (Go to http://www.gap-system.org/ and follow the instructions.)

   Then use GAP to do the following problems.

   (a) Use the command "NumberSmallGroups(n);" to find the number of groups of orders 64, 128, 256 and 288.

   (b) Multiply the permutations $(1, 2, 3, 4, 5, 6, 7)$ and $(1, 2)$. Comment. (Is this the answer our textbook author, Rotman, would give? See the footnote on page 2 of our textbook.)

   (c) Use the command "SmallGroupsInformation(n);" to briefly describe the groups of order 8.

3. Given a field F and a polynomial $f \in F[x]$, we view $f : x \mapsto f(x)$ as a function from F into F. We say that f is a "permutation polynomial" if this function is a permutation.

   (a) Show that *every* polynomial of degree one is a permutation polynomial.

   (b) Show that if the characteristic of the field F is not 2 then *no* polynomial of degree two is a permutation polynomial.

   (c) Show that the composition of permutation polynomials is a permutation polynomial.

   (d) Given a finite field F, let G be the set of all permutations on the elements of F which can be represented by a permutation polynomial. Show that G is a group.

   (e) (Rotman, page 15, problem 1.29) Show that $f(x) = 4x^2 - 3x^7$ is a permutation polynomial for $F = Z_{11}$. Write this permutation as a product of disjoint cycles.

   (f) Suppose F has characteristic p. Which permutation is represented by the polynomial $f(x) = x^p$?

4. Find an interesting webpage on the topic of group theory. Write a paragraph review of the webpage and send the link and review to me by email.

   The reviewed webpage cannot be a webpage at Wikipedia or Wolfram's MathWorld. The reviewed webpage cannot be a webpage already mentioned on **Ken's Group Theory Notes** webpage. (*That* webpage will be updated regularly with student solutions to this problem – please keep up with those updates in order to avoid duplicating a webpage reviewed by another student.)

Comments on Assignment 1.

1. This should be a fun problem. Just play with permutation cycles.... For part (c), you might suppose that σ and τ commute. ($LCM(a, b)$ represents the Least Common Multiple of a and b.
2. *GAP* is powerful software – and it is free! You don't need to be able to "program" in GAP to succeed in this course, but you should learn some very simple *GAP* commands.
3. (a) Straightforward.
   (b) Remember the quadratic formula.
   (c) Straightforward "definition tracing."
   (d) Straightforward, but good practice on using the group axioms.
   (e) Straightforward. Write out a two-row table with the domain $x = 0$ through 10 on the first row and the images $f(x)$ on the second row ...and just do the computations modulo 11. Then figure out the associated cycle decomposition.
   (f) Look up "Fermat's Little Theorem."
4. I hope this problem is fun. "Google" is a powerful tool.